DMAC 2469
DMPC 2469
DMSU 2469

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.1

Honour School of Mathematics and Philosophy Part C: Paper C7.1

Honour School of Mathematics and Statistics Part C: Paper C7.1

RELATIVITY

Trinity Term 2006

Monday 5 June 2006, 9.30am to 12.30pm

You may attempt as many questions as you like but **only your two best answers from each section of the paper will count**.

Start a new booklet for each question which you attempt. Hand in answers to each section in separate bundles. Indicate on the front sheet of each bundle the numbers of the questions attempted in that section. A booklet with the front sheet completed must be handed in for each section even if no question has been attempted in that section.

Do not turn this page until you are told that you may do so

A. Relativity I

1. A Flat Earth space-time has co-ordinates (t, x, y, z), where $z \geqslant 0$, and a metric

$$\mathrm{d}s^2 = (1 + gz)^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

where g is a positive constant.

Write down the geodesic equations in this space-time. Hence, or otherwise, show:

1. A Flat Earth physicist, stationary at a point with $z = h$, will measure a 'gravitational acceleration' of magnitude $g(1 + gh)^{-1}$.
2. A Flat Earth astronomer, stationary at $x = y = 0$ on the Flat Earth surface $z = 0$, and looking upwards at an angle θ to the Flat Earth surface, will see points of the Flat Earth surface with co-ordinates (x, y) such that

$$\sqrt{x^2 + y^2} = 2g^{-1} \tan\theta \, .$$

2. Suppose x^a, for $a = 0, 1, 2, 3$, are co-ordinates for an open region of a space-time and that O is the point where $x^a = 0$. Let g_{ab} be a metric, and define new co-ordinates by $\tilde{x}^a = x^a + \frac{1}{2}\Gamma^a{}_{bc} x^b x^c$ where $\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc})$. Show that in the new co-ordinates, the metric $\tilde{g}_{ab}$ has the property that

$$\frac{\partial \tilde{g}_{ab}}{\partial \tilde{x}^c} = 0 \text{ at } O.$$

We shall refer to such co-ordinates as *locally constant at O*. State the further conditions on the $\tilde{x}^a$ that must be satisfied for them to be local inertial co-ordinates at O.

Now consider the space-time metric

$$\mathrm{d}s^2 = (1 - 2m/r)\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2)$$

in the region $(-\infty < v < \infty,\ r > 0,\ 0 < \theta < \pi,\ -\pi < \phi < \pi)$. Let A be the point given by $(v = 0,\ r = 2m,\ \theta = \pi/2,\ \phi = 0)$. Show that the following new co-ordinates are locally constant at A:

$$\begin{aligned}
\tilde{x}^0 &= v + v^2/8m - m(\cos^2\theta + \sin^2\phi)\,, \\
\tilde{x}^1 &= (r - 2m)(1 - v/4m)\,, \\
\tilde{x}^2 &= -r\cos\theta\,, \\
\tilde{x}^3 &= r\sin\phi\,.
\end{aligned}$$

Find an example of local inertial co-ordinates at A, and state the freedom that you have in your choice.

3. Define the *tensor transformation law* for a tensor $T^a{}_b$ of type $(1,1)$.

State the general rule defining the *covariant derivative* of a tensor of type (p,q) with respect to a connection $\Gamma^a{}_{bc}$.

Now assume the connection to be the Levi-Civita connection

$$\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) \, .$$

Let X^a be an arbitrary contravariant vector field. Using the rule for covariant differentiation, show that the equation

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^d = R_{abc}{}^d X^c$$

is consistent and defines a tensor R_{abcd} independent of X^a. By expressing R_{abcd} in local inertial co-ordinates, show that

$$R_{abcd} = R_{ab[cd]} = R_{[ab]cd} = R_{cdab} \text{ and } R_{[abc]d} = 0 \, .$$

Explain carefully why it is sufficient to show that these symmetries hold in local inertial co-ordinates. Define the *Ricci tensor* R_{ab} and the *scalar curvature* R. Now define

$$C_{ab}{}^{cd} = R_{ab}{}^{cd} - 2R_{[a}{}^{[c} g_{b]}{}^{d]} + \tfrac{1}{3} R g_{[a}{}^c g_{b]}{}^d \, .$$

Show that C_{abcd} has the same symmetries as R_{abcd} and that additionally, $C_{cb}{}^{cd} = 0$.

4. Write down equations for time-like geodesics in the Schwarzschild metric:

$$\mathrm{d}s^2 = (1 - 2m/r)\mathrm{d}t^2 - (1 - 2m/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

where $(r > 2m)$. Show that there are geodesics satisfying $\theta \equiv \pi/2$, and explain why it is sufficient to consider such geodesics.

Show that by eliminating t and ϕ the equation of such a geodesic can be written in the form:

$$\dot{r}^2 = (E^2 - 1) + 2m/r - J^2/r^2 + 2mJ^2/r^3$$

where E and J are constants of the motion.

A particle is released into free fall at a point where $r = a$, with $\dot{r} = \dot{\theta} = \dot{\phi} = 0$ initially. Show that its equation of motion can be written as:

$$\dot{r}^2 = 2m(1/r - 1/a) \text{ for } r > 2m\,.$$

Show that the particle approaches the event horizon $r = 2m$ in a finite proper time.

Now take new co-ordinates given by

$$v = t + r + 2m\log(r - 2m),\ \rho = r,\ \theta' = \theta,\ \phi' = \phi\,.$$

Express the metric in these co-ordinates. Hence, show that the original region naturally extends to a larger space, with a metric which is regular at the points of the event horizon.

Show that the equation of the geodesic in the new co-ordinates is

$$\dot{\rho}^2 = 2m(1/\rho - 1/a) \text{ for } \rho > 0\,.$$

Hence show that the total proper time along the trajectory of the particle, between its release and its meeting the singularity $\rho = 0$, is

$$\frac{\pi}{2}\sqrt{\frac{a^3}{2m}}\,.$$

B. Relativity II

5. Given smooth vector fields X^a, Y^a and a smooth function f in the space-time M, define the *Lie derivatives* of f and Y^a along X^a. Define what is meant by saying that X^a is a *Killing vector*, and that it is *hypersurface orthogonal*.

Let ∇_a be the metric-covariant derivative in M. From the definitions of the Lie derivatives of f and Y^a along X^a above and the Leibniz rule, derive the following expression for the Lie derivative of an antisymmetric, smooth tensor field F_{ab} along X^a:

$$\mathcal{L}_X F_{ab} = X^c \nabla_c F_{ab} + F_{ac} \nabla_b X^c + F_{cb} \nabla_a X^c \ .$$

Suppose that F_{ab} satisfies the vacuum Maxwell equations

$$\nabla^a F_{ab} = 0 \qquad \text{and} \qquad \nabla_{[a} F_{bc]} = 0 \ ,$$

and that the Lie derivative of F_{ab} along X^a is zero. Show that, in a simply connected region, there exists a smooth function ϕ such that

$$F_{ab} X^b = \nabla_a \phi \ .$$

[*You may assume without proof that, in a simply connected region, for any co-vector field V_a satisfying $\partial_{[a} V_{b]} = 0$, there is a smooth function ψ such that $V_a = \partial_a \psi$.*]

Now suppose that X^a is a timelike hypersurface-orthogonal Killing vector. Show that

$$\nabla_a X_b = (\nabla_a U) X_b - (\nabla_b U) X_a \ ,$$

where $X^a X_a = \mathrm{e}^{2U}$. [*You may use without proof standard results for Killing vectors and hypersuface orthogonal vectors as long as they are stated clearly.*] Show that in this case, ϕ satisfies the equation

$$\nabla^a (\mathrm{e}^{-2U} \nabla_a \phi) = 0 \ ,$$

as a result of the Maxwell equations.

6. Let

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} \,,$$

be the metric on a space-time M, where η_{ab} is the standard Minkowski metric, ϵ is a small real constant, and h_{ab} is a symmetric smooth tensor field. Show that, when terms of order two and higher in ϵ are neglected, the inverse metric is

$$g^{ab} = \eta^{ab} - \epsilon\, h^{ab} \ ,$$

where $h^{ab} = \eta^{ac}\eta^{bd}h_{cd}$. Show that, to the same order in ϵ, the Christoffel symbols derived from g_{ab} are given by

$$\Gamma^a_{bc} = \frac{1}{2}\epsilon\eta^{ad}\left[\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right] \ .$$

What is the *de Donder* gauge condition?

Show that the linearised Einstein equations, with the de Donder gauge condition imposed and a source defined by the energy-momentum tensor T_{ab}, are

$$\epsilon \,\Box\, h_{ab} = -16\pi G \left(T_{ab} - \frac{1}{2}\eta_{ab}\,T\right) \ ,$$

where $\Box = \partial^a\partial_a$ and $T = \eta^{ab}T_{ab}$.

Now suppose that T_{ab} and the field h_{ab} are independent of time, and that T_{ab} is zero outside a spatially-bounded region. Show that, as a consequence of the conservation equation satisfied by T_{ab}, the integral of $T_{11} + T_{22} + T_{33}$ over a hypersurface of constant x^0 is zero. [*You could consider integrating* $\partial^i(T_{ij}x^j)$.]

Find an expression for h_{00} in terms of the total mass of the source which is correct to $\mathcal{O}(1/r)$ in terms of spatial distance r.

[*The following formulae may be of use:*

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right) \ , \qquad R_{abc}{}^d = 2\partial_{[a}\Gamma^d_{b]c} + 2\Gamma^e_{c[a}\Gamma^d_{b]e} \ .]$$

7. For a certain space-time with coordinates x^a, $a = 0, 1, 2, 3$, let the metric be $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$. If u is one of these coordinates, and the metric is such that none of its components g_{ab} depend on u, show that $V = \partial/\partial u$ is a Killing vector.

The Kerr solution can be written in Boyer–Lindquist coordinates (t, r, θ, ϕ) as

$$\begin{aligned} \mathrm{d}s^2 \;=\; & \left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4Mar\sin^2\theta}{\Sigma}\mathrm{d}t\,\mathrm{d}\phi - \frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\mathrm{d}\theta^2 \\ & - \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\mathrm{d}\phi^2 \ , \end{aligned}$$

where $\Delta = r^2 + a^2 - 2Mr$, $\Sigma = r^2 + a^2\cos^2\theta$, and $M > a \geqslant 0$ are constant parameters, is a solution of the Einstein field equations, describing a stationary axisymmetric body with mass M and angular momentum $J = Ma$. Deduce that the vector fields K and L defined by

$$K = K^a\nabla_a = \frac{\partial}{\partial t} \ , \qquad L = L^a\nabla_a = \frac{\partial}{\partial \phi} \ ,$$

are Killing vector fields. Do K and L commute? Is it possible for the vector field K to be hypersurface orthogonal? [*You may use without proof standard results for hypersuface orthogonal vectors as long as they are stated clearly.*] On which surfaces does the vector field K become null?

What does it mean for a hypersurface of constant r to be *null*? Find the values of r, r_+ and r_- with $r_+ > r_-$ on which the hypersurfaces of constant r are null. What does it mean for $r = r_+$ to be an *event horizon*?

8. The Robertson–Walker metric can be given as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R(r)^2 \left(\frac{\mathrm{d}\rho^2}{1 - k\rho^2} + \rho^2 (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \right) ,$$

where $k = 0, \pm 1$. What is the significance of k?

A cosmological model has the Robertson–Walker metric and a perfect fluid source. The Friedman equations for the model are

$$\dot{R}^2 + k = \frac{8\pi G}{3} \varrho R^2 , \qquad \dot{\varrho} + 3(\varrho + p)\frac{\dot{R}}{R} = 0 ,$$

where ϱ is the density of the fluid and p is the pressure.

Show that the Friedman equations are satisfied for $k = 0$ by $R = \mathrm{e}^{\alpha t}$ where α is a constant, if ϱ is a constant which you should find. What is the equation of state? Show that in this case, the Ricci tensor R_{ab} and the stress-energy-momentum tensor T_{ab} satisfy $R_{ab} = \Lambda g_{ab}$ and $8\pi G T_{ab} = \Lambda g_{ab}$ respectively, for some constant Λ which you should find. [*You may use the fact that the stress-energy-momentum tensor for a perfect fluid is* $T_{ab} = (p + \varrho)u_a u_b - p g_{ab}$.]

For this Friedman–Robertson–Walker model with $k = 0$ and $R = \mathrm{e}^{\alpha t}$, write the equation for the past light cone of the origin at time t_0, and show that events beyond a certain surface are never seen by an observer at the origin.

What is the *area distance* d_A of the galaxy as seen from the observer at the origin and time t_0? Obtain an expression for d_A in terms of $R(t_1)$ and $R(t_0)$. Given that the light from the galaxy is received with a cosmological red-shift z given by

$$z = \frac{R(t_0) - R(t_1)}{R(t_1)} ,$$

show that

$$d_A(z) = \frac{z}{H(1+z)} ,$$

where H is a constant which you should find.