Moderations Syllabus and Synopses 2003–2004 for examination in 2004

Contents

1 Foreword

Syllabus

The syllabus here is that referred to in the *Examination Regulations 2003*[1] and has been approved by the Subfaculty Teaching Committee for examination in Trinity Term 2004.

Synopses

The synopses give some additional detail, and show how the material is split between the different lecture courses. They also include details of recommended reading.

Practical Work

The requirement in the *Examination Regulations* to pursue an adequate course of practical work will be satisfied by following the Maple course and submitting two Maple projects. Details about submission of these projects will be given in the Maple handbook.

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet should be addressed to the Academic Assistant in the Mathematical Institute.

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[1]

Examination Regulations 2003

Special Regulations for the Honour Moderations in Mathematics

A

The subjects of the examination shall be Mathematics and its applications. The syllabus and the number of papers shall be as prescribed by regulation from time to time by the Mathematical and Physical Sciences Board.

B

1. Each candidate shall offer four papers as follows:
   A. Pure Mathematics I
   B. Pure Mathematics II
   C. Applied Mathematics I
   D. Applied Mathematics II
2. The syllabus for each paper will be published by the Mathematical Institute in a handbook for candidates by the beginning of the Michaelmas Full Term in the academic year of the examination, after consultation with the Subfaculty of Mathematics. Each paper will contain questions of a straightforward character.
3. The Chairman of Mathematics, or a deputy, shall make available to the moderators evidence showing the extent to which each candidate has pursued an adequate course of practical work. In assessing a candidate's performance in the examination, the moderators shall take this evidence into account.
4. No candidate shall be declared worthy of Honours who is not deemed worthy of Honours on the basis of papers A–D and no candidate shall be awarded a Pass who has not shown competence on these papers.
5. The use of calculators is generally not permitted but certain kinds may be allowed for certain papers. Specifications of which papers and which types of calculators are permitted for those exceptional papers will be announced by the examiners in the Hilary Term preceding the examination.

2 Syllabus

2.1 Pure Mathematics I

Naïve treatment of sets and mappings. Definition of a countable set. Relations, equivalence relations and partitions.

Vector spaces over $\mathbb{R}$; subspaces. The span of a (finite) set of vectors; spanning sets; examples. Finite dimensionality.

Linear dependence and linear independence. Definition of bases; reduction of a spanning set and extension of a linearly independent set to a basis; proof that all bases have the same size. Dimension of the space. Co-ordinates with respect to a basis.

Linear transformations from one (real) vector space to another. Rectangular matrices over $\mathbb{R}$. Algebra of matrices. The matrix representation of a linear transformation with respect to fixed bases; change of basis and co-ordinate systems. Composition of transformations and product of matrices.

Elementary row operations on matrices; echelon form and row-reduction. Invariance of the row space under row operations; row rank. Applications to finding bases of vector spaces.

Sums and intersections of subspaces; formula for the dimension of the sum.

The image and kernel of a linear transformation. The rank-nullity theorem. Applications.

Matrix representation of a system of linear equations. Significance of image, kernel, rank and nullity for systems of linear equations. Solution by Gaussian elimination. Bases of solution space of homogeneous equations.

Invertible matrices; use of row operations to decide invertibility and to calculate inverse. Column space and column rank. Equality of row rank and column rank.

Determinants of square matrices; properties of the determinant function. Computation of determinant by reduction to row echelon form. Relationship of determinant to invertibility. Determinant of a linear transformation of a vector space to itself.

Eigenvalues of linear transformations of a vector space to itself. The characteristic polynomial of a square matrix; the characteristic polynomial of a linear transformation of a vector space to itself. The linear independence of a set of eigenvectors associated with distinct eigenvalues; diagonalisability of matrices.

Division algorithm and Euclid's algorithm for integers. Polynomials with real or complex coefficients, factorization; Division algorithm and Euclid's algorithm for polynomials.

Permutations; composition of permutations. Cycles, the standard cycle-product notation. The parity of a permutation; how to calculate parity from cycle-structure.

Axioms for a group. Examples, including the symmetric groups. Subgroups, intersections of subgroups. The order of a subgroup. The order of an element. Cyclic subgroups. Cosets and Lagrange's Theorem. Straightforward examples. The structure of cyclic groups; orders of elements in cyclic groups.

Homomorphisms of groups. The (First) Isomorphism Theorem.

Euclidean geometry in two and three dimensions approached by vectors and co-ordinates.

Vector algebra. Equations of planes, lines and circles. Rotations, reflections, isometries. Parametric representation of curves, tangents, arc-length (not Serret–Frenet formulae). Simple surfaces: spheres, right circular cones. Gradient as normal vector to a surface.

2.2 Pure Mathematics II

Real numbers: arithmetic, ordering, suprema, infima; real numbers as a complete ordered field. The reals numbers are uncountable.

The complex number system. The triangle inequality.

Sequences of (real or complex) numbers. Limits of sequences of numbers; the algebra of limits. Order notation.

Subsequences; every subsequence of a convergent sequence converges to the same limit. Bolzano–Weierstrass Theorem. Cauchy's convergence principle. Limit point of a subset of the line or plane.

Series of (real or complex) numbers. Convergence of series. Simple examples to include geometric progressions and power series. Alternating series test, absolute convergence, comparison test, ratio test, integral test.

Power series, radius of convergence, important examples including exponential, sine and cosine series.

Continuous functions of a single real or complex variable. The algebra of continuous functions. A continuous real valued function on a closed bounded interval is bounded, achieves its bounds and is uniformly continuous. Intermediate value theorem. Inverse Function Theorem for continuous strictly monotonic functions.

Sequences and series of functions. The uniform limit of a sequence of continuous functions is continuous. Weierstrass's M-test. Continuity of functions defined by power series.

Definition of derivative of a function of a single real variable. The algebra of differentiable functions. Rolle's Theorem. Mean Value Theorem. L'Hôpital's rule. Taylor's expansion with remainder in Lagrange's form. Binomial theorem with arbitrary index.

Step functions and their integrals. The integral of a continuous function of a closed bounded interval defined as the supremum of the integral of the step functions it dominates. Properties of the integral including linearity and the interchange of integral and limit for a uniform limit of continuous functions on a bounded interval. The Mean Value Theorem for Integrals. The Fundamental Theorem of Calculus; integration by parts and substitution.

Term by term differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly); relationships between exponential, trigonometric functions and hyperbolic functions.

The geometry of the complex numbers. The Argand diagram. Complex numbers as a method of studying plane geometry, roots of unity, cocyclic points, Ptolemy's Theorem, equations of lines and circles (Apollonius's Theorem). Möbius transformations acting on the extended complex plane. Möbius transformations take lines and circles to lines and circles. The Riemann sphere and the conformal map to the extended complex plane.

The geometry of the sphere: great circles, triangles (their six angles, angle-excess and area formula).

Platonic solids.

2.3 Applied Mathematics I

Standard integrals, integration by parts. General linear homogeneous ordinary differential equations: integrating factor for first order linear ordinary differential equations, second solution when one solution known for second order linear ordinary differential equations. First and second order linear ordinary differential equations with constant coefficients. General solution of linear inhomogeneous ordinary differential equation as particular solution plus solution of homogeneous equation. Simple examples of finding particular integrals by guesswork.

Systems of linear coupled first order ordinary differential equations.

Calculation of determinants, eigenvalues and eigenvectors.

Newton's laws. Free and forced linear oscillations. Simple oscillatory systems with two degrees of freedom, natural frequencies. Two dimensional motion, projectiles. Use of polar coordinates, circular motion. Central forces, differential equation for the particle path. Inverse square law, planetary orbits.

Energy and potential for one dimensional motion. Equivalent ideas for central force problems and three dimensional problems with axial symmetry.

Examples of stability and instability in physical situations, via linearlzed equations. Simple ideas of phase space, stable and unstable fixed points, periodic orbits. Informal introduction to chaos.

Sample space as the set of all possible outcomes, events and probability function. Permutations and combinations, examples using counting methods, sampling with or without replacement. Algebra of events. Conditional probability, partitions of sample space, theorem of total probability, Bayes's theorem, independence. Examples with statistical implications.

Random variable. Probability mass function. Discrete distributions: Bernoulli, binomial, Poisson, geometric, situations in which these distributions arise. Expectation: mean and variance. Probability generating functions, use in calculating expectations. Bivariate discrete distribution, conditional and marginal distributions. Extensions to many random variables. Independence for discrete random variables. Random walks (finite state space only). Solution of quadratic difference equations with applications to random walks.

Expectations of functions of more than one random variable. Random sample. Conditional expectation, application of theorem of total probability to expectation of a random variable. Sums of independent random variables. Examples from well-known distributions.

Continuous random variables. Cumulative distribution function for both discrete and continous random variables. Probability density function. Examples: uniform, exponential, gamma, normal. Practical examples. Expectation. Cumulative distribution function and probability density function for a function of a single continuous random variable. Simple examples of joint distributions of two or more continuous random variables, independence, expectation (mean and variance of sums of independent, identically distributed random variables).

Random sample, concept of a statistic and its distribution, sample mean and sample variance.

Concept of likelihood, examples of likelihood for simple distributions. Estimation for a single unknown parameter by maximising likelihood. Examples drawn from: Bernoulli, binomial, geometric, Poisson, exponential (parametrised by mean), normal; (mean only, variance known). Data to include simple surveys, opinion polls, archaeological studies. Properties of estimators: unbiasedness, Mean Squared Error =((bias)^2+ variance). Statement of Central Limit Theorem (excluding proof). Confidence intervals using Central Limit Theorem. Simple straight line fit, $Y_t = a + bx_t + \epsilon_t$, with ϵ_t normal independent errors of zero mean and commom known variance; estimators for a, b by maximising likelihood using partial differentiation, unbiasedness and calculation of variance as linear sums of Y_t. (No confidence intervals). Examples (use scatterplots to show suitability of linear regression).

2.4 Applied Mathematics II

Functions of several real variables: continuity, partial derivatives, chain rule, change of variable, calculation of areas, double integrals (informal treatment), evaluation by change of variable. C^n functions. Statement of conditions for equality of mixed partial derivatives. Statement of Taylor's theorem for a function of two variables. Critical points. Gradient vector. Informal (geometrical) treatment of Lagrange multipliers.

Elementary partial differential equations. Introduction to Laplace's equation, Poisson's equation, the wave equation and the heat equation. Verification of solutions to these equations.

Fourier series. Periodic, odd and even functions. Calculation of sine and cosine series. Simple applications (concentrating on the calculation of Fourier coefficients and the use of Fourier series).

Derivation of (i) the wave equation of a string, (ii) the heat equation in one and two dimensions (box argument only). Examples of solutions and their interpretation. Boundary conditions

Use of Fourier series to solve the wave equation, Laplace's equation and the heat equation (all in two independent variables) Applications. D'Alembert's solution of the wave equation and applications. Characteristic diagrams (excluding reflection and transmission). Transformations in the independent variables. Solution by separation of variables.

Uniqueness theorems for the wave equation, heat equation and Laplace's equation (all in two independent variables). Energy.

Integrals along curves in the plane. Green's theorem in the plane (informal proof only).

Div, grad and curl in Euclidean coordinates. Evaluation of line, surface and volume integrals. Stokes' theorem and the Divergence theorem in two and three variables (proof excluded).

Rederivation of models of continuity of flow: the heat equation from the Divergence theorem.

Gravity as a conservative force. Gauss's theorem. The equivalence of Poisson's equation and the inverse-square law.

3 Synopses of Lectures

3.1 Introductory Mathematics

3.1.1 Introductory Mathematics — Prof Batty — 5 MT

5 introductory lectures in the first week of Michaelmas term

The purpose of these introductory lectures is to establish some of the basic notation of mathematics, introduce the elements of (naïve) set theory and the nature of formal proof.

Synopsis

Sets: examples including the natural numbers, the integers, the rational numbers, the real numbers; inclusion, union, intersection, power set, ordered pairs and cartesian product of sets.

Maps: composition, restriction, 1–1ness, ontoness, invertible maps, images and preimages.

Definition of a countable set. The countability of the rational numbers.

The well-ordering property of the natural numbers. Induction as a method of proof, including a proof of the binomial theorem with integer coefficients.

Reading

1. G. Smith, *Introductory Mathematics: Algebra and Analysis*, Springer-Verlag (1998), Chapters 1 and 2.
2. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis*, Third Edition (2000), Wiley, Chapter 1 and Appendices A and B.
3. C. Plumpton, E. Shipton, R. L. Perry, *Proof*, MacMillan (1984).

3.2 Pure Mathematics I

3.2.1 Linear Algebra — Dr Knight & Dr Scataglini — 14 MT + 8 HT

Linear algebra pervades and is fundamental to geometry (from which it originally arose), algebra, analysis, applied mathematics, statistics—indeed all of mathematics.

The course has several aims. The first is to introduce students through a thorough study of two- and three-dimensional spaces to the general concept of a vector space, subspaces, and the ideas of linear dependence, independence, spanning sets, bases, dimension.

A second aim is to introduce students to matrices and their applications to the algorithmic solution of systems of linear equations and to the study of linear transformations of vector spaces.

A third aim is to introduce determinants and their properties. A fourth aim is to introduce eigenvalue theory and some of its applications.

Synopsis

Fourteen lectures in Michaelmas Term

Introduction: examples of linear problems (*e.g.*, system of linear equations, differential equations) and their solutions. Vectors in the plane and 3-space, and co-ordinates. Addition of vectors and multiplication of vectors by scalars corresponding to co-ordinatewise operations in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Linear combinations of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Lines and planes as subspaces of $\mathbb{R}^2$, $\mathbb{R}^3$. The subspace spanned by a set of vectors.

Linear independence of vectors in $\mathbb{R}^2$, $\mathbb{R}^3$. Bases; co-ordinates with respect to a basis; co-ordinate changes. Testing for linear independence (introducing and using 2×2 and 3×3 matrices).

Linear transformations of $\mathbb{R}^2$, $\mathbb{R}^3$, with geometric examples. Specification of such transformations by matrices. Examples of simplification by good choice of basis. Kernels and images.

Definition of a vector space (over $\mathbb{R}$; brief mention of $\mathbb{C}$ and $\mathbb{Q}$); examples (*e.g.*, $\mathbb{R}^n$, polynomials). Simple consequences of the axioms. Subspaces; examples.

The span of a (finite) set of vectors; spanning sets; examples. Finite dimensionality [all spaces should be finite-dimensional for the rest of the course].

Linear dependence and linear independence. Definition of a basis; co-ordinates with respect to a basis.

Linear transformations from one (real) vector space to another; examples. Rectangular matrices over $\mathbb{R}$ (brief mention of $\mathbb{C}$ and $\mathbb{Q}$); row and column vectors as matrices. For given bases, correspondence between linear transformations and matrices. Sums, scalar multiples. Composition of transformations and product of matrices.

Reduction of a spanning set and extension of a linearly independent set to a basis; proof that all bases have the same size. Dimension of a space.

Elementary row operations on matrices; echelon form and row-reduction. Invariance of the row space under row operations; row rank. Applications to finding bases.

Sums and intersections of subspaces; formula for the dimension of the sum.

The image and kernel of a linear transformation. The rank-nullity theorem. Applications.

Matrix representation of a system of linear equations. Significance of image, kernel, rank and nullity for systems of linear equations. Solution by Gaussian elimination. Bases of solution space of homogeneous equations.

Invertible matrices; use of row operations to decide invertibility and to calculate inverse. Column space and column rank. Equality of row rank and column rank.

Eight lectures in Hilary Term

The matrix of a linear transformation with respect to bases, and change of bases.

Determinants of square matrices [facts about permutations and parity to be stated and used as necessary—proofs to be given later in the term]; properties of the determinant function; determinants and the scalar triple product. Computation of determinant by reduction to row echelon form. Proof that a square matrix is invertible if and only if it is non-singular. Determinant of a linear transformation of a vector space to itself.

Eigenvalues of linear transformations of a vector space to itself. The characteristic polynomial of a square matrix; the characteristic polynomial of a linear transformation of a vector space to itself. The linear independence of a set of eigenvectors associated with distinct eigenvalues; diagonalisability of matrices.

Reading List

1. C. W. Curtis, *Linear Algebra—an Introductory Approach*, Springer (4th edition), reprinted 1994
2. R. B. J. T. Allenby, *Linear Algebra*, Arnold, 1995
3. T. S. Blyth and E. F. Robertson, *Basic Linear Algebra*, Springer, 1998
4. D. A. Towers, *A Guide to Linear Algebra*, Macmillan, 1988
5. D. T. Finkbeiner, *Elements of Linear Algebra*, Freeman, 1972 [Out of print, but available in many libraries]
6. B. Seymour Lipschutz, Marc Lipson, *Linear Algebra* Third Edition 2001

3.2.2 Some Theory of Sets and Groups — Dr Stoy — 8 HT, 8 TT

Abstract algebra evolved in the twentieth century out of nineteenth century discoveries in algebra, number theory and geometry. It is a highly developed example of the power of generalisation and axiomatisation in mathematics.

The first objective is to develop set theory as a useful language for abstract mathematics. Next we aim to introduce the student to some of the basic techniques of algebra. Thirdly, we aim to introduce group theory through axioms and examples both because groups are typical and famous examples of algebraic structures, and because of their use in the measurement of symmetry.

Synopsis

Eight lectures in Hilary Term

Relations, equivalence relations and partitions; examples.

Division algorithm and Euclid's algorithm for integers. Polynomials with real or complex coefficients, factorization; Division algorithm and Euclid's algorithm for polynomials; the Remainder Theorem.

Permutations; composition of permutations. Cycles, the standard cycle-product notation. The parity of a permutation; the goodness of the definition; how to calculate parity from cycle-structure.

Eight lectures in Trinity Term

The symmetric groups. Further examples, both arithmetic and geometric, to motivate the study of groups.

The concept of a binary operation. Axioms for a group. Subgroups. Intersections of subgroups.

The order of a subgroup. The order of an element. Cyclic subgroups. Cosets and Lagrange's Theorem. Straightforward examples. The structure of cyclic groups; orders of elements in cyclic groups.

Motivating examples of homomorphisms; comparison with linear transformations. Kernels, normal subgroups and quotient groups. The (First) Isomorphism Theorem.

Reading List

Peter J. Cameron, *Introduction to Algebra*, (OUP 1998), §§1.1, 1.2, 1.3, 1.4, 3.1, 3.2, 3.3, 3.4.

Alternative reading

1. G. C. Smith, *Introductory Mathematics: Algebra and Analysis*, Springer Undergraduate Mathematics Series 1998, Chapters 1, 2, 5.
2. Joseph J. Rotman, *A first course in abstract algebra* (Second edition, Prentice-Hall 2000), §§1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.3, 3.5.
3. P. M. Cohn, *Classic Algebra*, (Wiley 2000), Chapters 1, 2, 3.
4. I. N. Herstein, *Topics in algebra* (Second edition, Wiley 1975), §§1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 2.4, 2.6, 2.7, 2.10

3.2.3 Geometry I — Dr Howison — 7 MT

These lectures give an introduction to elementary ideas in the geometry of Euclidean space through vectors.

Synopsis

Euclidean geometry in two and three dimensions approached by vectors and co-ordinates. Vector addition and scalar multiplication. The scalar product, equations of planes, lines and circles. The vector product in three dimensions. Scalar triple products and vector triple products, vector algebra. Rotations, reflections, isometries. Parametric representation of curves, tangents, arc-length (but not Serret–Frenet formulae). Simple surfaces: spheres, right circular cones. Gradient as normal vector to a surface.

Reading

1. P. M. Cohn, *Solid Geometry*, Routledge and Kegan Paul.
2. J. Roe, *Elementary Geometry*, Oxford Science Publications (1992),Chapters 2.2, 3.4, 4,7.1, 7.2, 8.1, 8.2.
3. Jerrold E. Marsden and Anthony J. Tromka, *Vector Calculus*, McGraw–Hill, fourth edition(1996), Chapters 1,2.

3.3 Pure Mathematics II

Students will already be aware of the importance and power of calculus. The aim of Analysis I, II and III is to put precision into its formulation and to make students more secure in their understanding and use. The lectures on analysis come in three parts, beginning in the first term with the properties of the real numbers and studying limits through sequences and series. The second part is concerned with continuity and differentiation, and the third part with integration. Power series and the functions they define provide a unifying theme.

3.3.1 Analysis I: Sequences and series — Prof Batty — 14 MT

In these lectures we introduce the real and complex numbers and study their properties, particulary completeness; define and study limits of sequences, convergence of series, and power series.

Synopsis

Real numbers: arithmetic, ordering, suprema, infima; the real numbers as a complete ordered field. The reals are uncountable. The complex number system. The triangle inequality.

Sequences of real or complex numbers. Definition of a limit of a sequence of numbers. Limits and inequalities. Order notation: O, o.

The algebra of limits. Subsequences; a proof that every subsequence of a convergent sequence converges to the same limit; bounded monotone sequences converge. Bolzano–Weierstrass Theorem. Cauchy's convergence principle. Limit point of a set.

Series of real or complex numbers. Convergence of series. Simple examples to include geometric progressions and some power series. Alternating series test, absolute convergence, comparison test, ratio test, integral test.

Power series, radius of convergence; important examples to include the exponential, sine and cosine series.

Reading — Main texts

1. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis*, Third Edition(2000), Wiley. Chapters 2, 3, 9.1, 9.2.

2. R. P. Burn, *Numbers and Functions, Steps into Analysis*, Cambridge University Press (2000), Chapters 2–6. (This is a book of problems and answers, a DIY course in analysis.)

Alternative reading

The first five books take a slightly gentler approach to the material in the syllabus, whereas, the last two cover it in greater depth and contain some more advanced material.

1. Mary Hart, *A Guide to Analysis*, MacMillan (1990), Chapter 2.
2. J. C. Burkill, *A First Course In Mathematical Analysis*, CUP (1962), Chapters 1, 2 and 5.
3. K. G. Binmore, *Mathematical Analysis, a straightforward approach*, Cambridge University Press, Chapters, 1–6.
4. Victor Bryant, *Yet Another Introduction to Analysis*, Cambridge University Press (1990), Chapters 1 and 2.
5. G. Smith, *Introductory Mathematics: Algebra and Analysis*, Springer-Verlag (1998), Chapter 3 (introducing complex numbers).
6. Michael Spivak, *Calculus*, Benjamin (1967), Parts I, IV, and V (for a construction of the real numbers).
7. Brian S. Thomson, Judith B. Bruckner, Andrew M. Bruckner, *Elementary Analysis*, Prentice Hall (2001), Chapters 1–4.

3.3.2 Analysis II: Continuity and differentiability — Prof Heath-Brown — 16 HT

In this term's lectures we study continuity functions of a real or complex variable variable, and differentiablity of functions of a real variable.

Synopsis

Definition of $\lim_{x\to a} f(x)$. Examples and counter examples to illustrate when $\lim_{x\to a} f(x) = f(a)$ (and when it doesn't).

Definition of continuity of functions on subsets of $\mathbb{R}$ and $\mathbb{C}$ in terms of ϵ and δ. The algebra of continuous functions; examples, including polynomials.

Continuous functions on closed bounded intervals: boundedness, maxima and minima, uniform continuity.

Intermediate Value Theorem. Inverse Function Theorem for continuous strictly monotone function.

Sequences and series of functions. The uniform limit of a sequence of continuous functions is continuous. Weierstrass's M-test for uniformly convergent series of functions. Continuity of functions defined by power series.

Definition of the derivative of a function of a real variable. Algebra of derivatives, examples to include polynomials and inverse functions. Statement (proof deferred to TT) that the derivative of a function defined by a power series is given by the derived series.

Vanishing of the derivative at a local maximum or minimum. Rolle's Theorem. Mean Value Theorem with simple applications: constant and monotone functions.

Cauchy's (generalized) Mean Value Theorem and l'Hôpital's formula. Taylor's Theorem with remainder in Lagrange's form; examples of Taylor's Theorem to include the binomial expansion with arbitrary index.

Reading — Main texts

1. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis*, Third Edition (2000), Wiley. Chapters 4–8.
2. R. P. Burn, *Numbers and Functions, Steps into Analysis*, Cambridge University Press, 2000. This is a book of problems and answers, a DIY course in analysis. Chapters 6–9, 12.

Alternative reading

1. Mary Hart, *A Guide to Analysis*, MacMillan (1990), Chapters 4,5.
2. J. C. Burkill, *A First Course in Mathematical Analysis*, CUP (1962), Chapters 3, 4, and 6.
3. K. G. Binmore, *Mathematical Analysis, a straightforward approach*, CUP Chapters 7–12, 14–16.
4. Victor Bryant, *Yet Another Introduction to Analysis*, CUP (1990), Chapters 3 and 4
5. M. Spivak, *Calculus*, 3rd Edition, Publish or Perish (1994), Part III.
6. Brian S. Thomson, Judith B. Bruckner, Andrew M. Bruckner, *Elementary Analysis*, Prentice Hall (2001), Chapters 5–10.

3.3.3 Analysis III: Integration — Prof Lyons — 8 TT

In these lectures we define a simple integral and study its properties; prove the Mean Value Theorem for Integrals and the Fundamental Theorem of Calculus. This gives us the tools to justify term by term differentiation of power series and deduce the elementary properties of the trigonometric functions.

Synopsis

Step functions and their integral. The integral of a continuous function on a closed bounded interval defined as the supremum of the integral of the step functions it dominates. Elementary properties of the integral: positivity, subdivision of the interval.

The Mean Value Theorem for Integrals. The Fundamental Theorem of Calculus; linearity of the integral, integration by parts and substitution.

The interchange of integral and limit for a uniform limit of continuous functions on a bounded interval. Term by term integration and differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly); examples to include the derivation of the main relationships between exponential, trigonometric functions and hyperbolic functions.

Reading

1. J. Roe, *Integration*, Mathematical Institute Notes (1994).
2. H. A. Priestley, *Introduction to Integration*, Oxford Science Publications (1997), Chapters 1–8. (These chapters commence with a useful summary of background 'cont and diff' and go on to cover not only the integration but also the material on power series.
3. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis*, Third Edition (2000), Wiley, Chapter 8.

3.3.4 Geometry II — Prof Tod — 8 TT

The aim of these last 8 lectures in geometry is to introduce students to some of the most elegant and fundamental results in the geometry of the Euclidean plane and the sphere.

Synopsis

The geometry of the complex numbers. The Argand diagram. Complex numbers as a methods of studying plane geometry, examples to include roots of unity, cocyclic points, Ptolemy's Theorem. Equations of lines and circles (Apollonius's Theorem), Möbius transformations acting on the extended complex plane. Möbius transformations take lines and circles to lines and circles, The Riemann sphere and the conformal map to the extended complex plane.

The geometry of the sphere: great circles, triangles (their six angles, angle-excess and area formula).

Platonic solids and the corresponding subdivision of the sphere noting Euler's formula.

Reading

1. Roger Fenn, *Geometry*, Springer (2001), Chapters 4 and 8.
2. J. Roe, *Elementary Geometry*, Oxford Science Publications (1992), Section 6.5
3. Liang-shin Hahn, *Complex Numbers and Geometry*, The Mathematical Association of America (1994).
4. George A. Jennings, *Modern Geometry and Applications*, Springer (1994), Chapter 2.

5. David A. Brannan, Matthew F. Esplen, Jeremy Gray, *Geometry*, Cambridge University Press (1999), Chapter 7.

6. Elmer G. Rees, *Notes on Geometry*, Springer-Verlag (1983), pp 23-28.

3.4 Applied Mathematics I

3.4.1 Calculus of One Variable — Dr Laws — 6 MT

These lectures are designed to give students a gentle introduction to applied mathematics in their first term at Oxford, allowing time for both students and tutors to work on developing and polishing the skills necessary for the course. It will have an 'A-level' feel to it, helping in the transition from school to university. The emphasis will be on developing skills and familiarity with ideas using straightforward examples.

Background reading:

D. W. Jordan & P. Smith, *Mathematical Techniques*, 2nd Edition, Oxford (1997) [3rd Edition out soon] Chapters 1–4, 14–17.

Synopsis

Standard integrals, integration by parts.
Definition of order of an ODE - example of separation of variables.
General linear homogeneous ODEs: integrating factor for first order linear ODEs, second solution when one solution known for second order linear ODEs.
First and second order linear ODEs with constant coefficients.
General solution of linear inhomogeneous ODE as particular solution plus solution of homogeneous equation.
Simple examples of finding particular integrals by guesswork.
(Jordan & Smith, Chapters 18, 19, 22; Kreyszig, Sections 1.1–1.3, 1.6, 2.1–2.3, 2.8.)

Systems of linear coupled first order ODEs.
The calculation of determinants, eigenvalues and eigenvectors.
(Kreyszig, Sections 3.0–3.3.)

Reading

1. D. W. Jordan & P. Smith, *Mathematical Techniques*, 2nd Edition, Oxford (1997). [3rd edition due out soon]

2. Erwin Kreyszig, *Advanced Engineering Mathematics*, 8th Edition, Wiley (1999).

3.4.2 Dynamics — Dr Acheson — 16 MT

The subject of dynamics is about how things change with time. A major theme is the modelling of a physical system by differential equations, and one of the highlights involves using the law of gravitation to account for the motion of planets.

Synopsis

Newton's laws. Free and forced linear oscillations. Simple oscillatory systems with two degrees of freedom, natural frequencies. Two dimensional motion, projectiles. Use of polar coordinates, circular motion. Central forces, differential equation for the particle path. Inverse square law, planetary orbits.

Energy and potential for one dimensional motion. Equivalent ideas for central force problems and three dimensional problems with axial symmetry.

Examples of stability and instability in physical situations, via linearized equations. Simple ideas of phase space, stable and unstable fixed points, periodic orbits. Informal introduction to chaos.

Reading

> David Acheson, *From Calculus to Chaos: an introduction to dynamics*, OUP (1997), Chapters 1,5,6,10,11.

Further reading

1. M. W. McCall, *Classical Mechanics: a modern introduction*, Wiley, (2001). Chapters 1–4, 7.
2. M. Lunn, *A First Course in Mechanics*, OUP (1991), Chapters 1–3 (up to 3.4).

3.4.3 Probability — Prof Tod and Dr A D Lunn — 8 MT, 8 HT

An understanding of random phenomena is becoming increasingly important in today's world within social and political sciences, finances, life sciences and many other fields. The aim of this introduction to probability is to develop the concept of chance in a mathematical framework. Both discrete and continuous random variables are introduced, with examples involving most of the common distributions.

Synopsis

Michaelmas Term

Motivation, relative frequency, chance. (What do we mean by a 1 in 4 chance?) Sample space as the set of all possible outcomes—examples. Events and the probability function. Permutations and combinations, examples using counting methods, sampling with or without replacement. Algebra of events. Conditional probability, partitions of sample space, theorem of total probability, Bayes's theorem, independence. Examples with statistical implications.

Random variable. Probability mass function. Discrete distributions: Bernoulli, binomial, Poisson, geometric, situations in which these distributions arise. Expectation: mean and variance. Probability generating functions, use in calculating expectations. Bivariate discrete distribution, conditional and marginal distributions. Extensions to many random variables. Independence for discrete random variables. Random walks (finite state space only). Solution of quadratic difference equations with applications to random walks.

Hilary Term

Expectations of functions of more than one random variable. Random sample. Conditional expectation, application of theorem of total probability to expectation of a random variable. Sums of independent random variables. Examples from well-known distributions.

Continuous random variables, motivation. Cumulative distribution functions for both discrete and continous random variables. Probability density function – analogy with mass and density of matter. Examples: uniform, exponential, gamma, normal. Practical examples. Expectation. Cdf and pdf for function of a single continuous random variable. Simple examples of joint distributions of two or more continuous random variables; independence, expectation (mean and variance of sums of independent, identically distributed random variables).

Main reading

1. D. Stirzaker, *Elementary Probability*, CUP (1994), Chapters 1–4, 5.1–5.6, 6.1–6.3, 7.1, 7.2, 7.4, 8.1, 8.3, 8.5 (excluding the joint generating function).
2. D. Stirzaker, *Probability and Random Variables: A Beginner's Guide*, CUP (1999).

Further reading

1. J. Pitman, *Probability*, Springer-Verlag (1993).
2. S. Ross, *A First Course In Probability*, Prentice-Hall (1994).
3. G. R. Grimmett and D. J. A. Welsh, *Probability: An Introduction*, OUP (1986), Chapters 1–4, 5.1–5.4, 5.6, 6.1, 6.2, 6.3 (parts of), 7.1–7.3, 10.4.

3.4.4 Statistics — Prof Donnelly — 8 HT

The theme is the investigation of real data using the method of maximum likelihood to provide point estimation, given unknown parameters in the models. Maximum likelihood will be the central unifying approach. Examples will involve a distribution with a single unknown parameter, in cases for which the confidence intervals may be found by using the Central Limit theorem (statement only). The culmination of the course will be the link of maximum likelihood technique to a simple straight line fit with normal errors.

Synopsis

Random sample, concept of a statistic and its distribution, sample mean as a measure of location and sample variances a measure of spread.

Concept of likelihood– examples of likelihood for simple distributions. Estimation for a single unknown parameter by maximising likelihood. Examples drawn from: Bernoulli, Binomial, Geometric, Poisson, Exponential (parametrized by mean), Normal; (mean only, variance known). Data to include simple surveys, opinion polls, archaeological studies, etc. Properties of estimators—unbiasedness, Mean Squared Error =((bias)^2+ variance). Statement of Central Limit Theorem (excluding proof). Confidence intervals using CLT. Simple straight line fit, $Y_t = a + bx_t + \epsilon_t$, with ϵ_t normal independent errors of zero mean and commom known variance. Estimators for a, b by maximising likelihood using partial differentiation, unbiasedness and calculation of variance as linear sums of Y_t. (No confidence intervals). Examples (use scatterplots to show suitability of linear regression).

Reading

F. Daly, D. J. Hand, M. C. Jones, A. D. Lunn, K. J. McConway, *Elements of Statistics*, Addison Wesley, (1995). Chapters 1–5 give background including plots and summary statistics, Chapter 6 and parts of Chapter 7 are directly relevant.

Further reading

J. A. Rice, *Mathematical Statistics and Data Analysis*, Wadsworth and Brooks Cole, (1988).

3.5 Applied Mathematics II

3.5.1 Calculus of Two or More Variables — Dr Laws — 10 MT

Synopsis

10 lectures Michaelmas Term continuing on from ODEs

Introduction to partial derivatives.
Chain rule, change of variable.
Jacobians for two variable systems, calculations of areas.
Gradient vector.
(Jordan & Smith, Chapters 27–29, 31. Kreyszig appendix 3.2, Sections 8.8, 8.9, 9.3, 9.5, 9.6. Bourne & Kendall, Sections 4.1–4.5.)

Elementary PDEs with motivation for where they arise in applications.
Introduction to Laplace's equation, Poisson's equation, the wave equation and the heat equation.
Verification of solutions to these equations.
(Kreyszig, Sections 11.1, 11.2; Acheson, Chapter 7.)

Reading

1. D. W. Jordan & P. Smith, *Mathematical Techniques*, 2nd Edition, Oxford (1997). [3rd edition due out soon]

2. Erwin Kreyszig, *Advanced Engineering Mathematics*, 8th Edition, Wiley (1999).
3. D. E. Bourne & P. C. Kendall, *Vector Analysis and Cartesian Tensors*, Stanley Thornes (1992).
4. David Acheson, *From Calculus to Chaos: an introduction to dynamics*, OUP (1997).
5. G. F. Carrier & C. E. Pearson, *Partial Differential Equations — Theory and Technique*, Academic Press (1988). [Advanced — leads on to 2nd year PDEs]

3.5.2 Fourier Series and Two Variable Calculus — Dr Dyson — 16 HT

The first half of these lectures introduce students to Fourier series, concentrating on their practical application rather than proofs of convergence.

The second half deals with scalar functions of two independent variables. Students will learn how to evaluate area and line integrals and how they are related via Green's theorem. They will also be introduced to the ideas of continuity and differentiability of two-variable functions and shown how classify critical points using Taylor's theorem.

Synopsis

Fourier series. Periodic, odd and even functions. Calculation of sine and cosine series.
Simple applications, concentrating on imparting familiarity with the calculation of Fourier coefficients and the use of Fourier series.
Issues of convergence addressed through example and special cases.
No attempt to state or prove convergence theorems, though students should be aware that convergence is linked to smoothness, referring to Maple exercises, e.g., exploring the Gibbs phenomenon.
(Jordan & Smith, Chapter 26. Kreyszig, Sections 10.1–10.4.)

Informal definition of double integrals. Evaluation by change of variable.
Simple applications.
(Jordan & Smith, Chapters 29, 31. Kreyszig, Sections 8.8, 9.3, 9.5, 9.6.)

Integrals along curves in the plane.
Green's theorem in the plane *(informal proof only).*
(Jordan & Smith, Chapter 32. Kreyszig, Sections 9.1, 9.4.)

Definitions of continuity, of partial derivatives and the gradient vector in terms of limits.
C^n functions.
Conditions for equality of mixed partial derivatives.
Taylor's theorem for a function of two variables *(statement only).*
Critical points.
(J. C. Burkill, Chapter 8.)

Reading

1. D. W. Jordan & P. Smith, *Mathematical Techniques*, 2nd Edition, Oxford (1997). [3rd edition due out soon]

2. Erwin Kreyszig, *Advanced Engineering Mathematics*, 8th Edition, Wiley (1999).

3. J. C. Burkill, *A First Course in Mathematical Analysis*, CUP(Reprinted 1991), Chapter 8.

3.5.3 Partial Differential Equations in Two Dimensions and Applications — Dr Acheson — 16 HT

In these lectures, students will be shown how the heat equation, the wave equation and Laplace's equation arise in physical models. They will learn basic techniques for solving each of these equations in two independent variables, and will be introduced to elementary uniqueness theorems.

Synopsis

Introductory lecture in descriptive mode on particular differential equations and how they arise.
Derivation of (i) the wave equation of a string, (ii) the heat equation in one and two dimensions *(box argument only)*.
Examples of solutions and their interpretation. Boundary conditions.
(Kreyszig,Sections 11.1, 11.2. Carrier & Pearson, Sections 1.1, 1.2, 3.1, 3.2, 4.1, 4.2. Strauss, Sections 1.3, 1.4.)

Use of Fourier series to solve the wave equation, Laplace's equation and the heat equation *(all with two independent variables)*. Applications.
D'Alembert's solution of the wave equation and applications. Characteristic diagrams *(excluding reflection and transmission)*.
Transformations in the independent variables.
Solution by separation of variables.
(Kreyszig, Sections 11.3–11.5. Carrier & Pearson, Sections 1.3–1.8, 3.3–3.6, 4.4, 4.5. Strauss, Sections 2.1, 5.1, 5.2, 6.2, chapter 4.)

Uniqueness theorems for the wave equation, heat equation and Laplace's equation *(all in two independent variables)*. Energy.
(Carrier & Pearson, Sections 1.9, 4.3. Strauss, Sections 2.2, 2.3, 6.1.)

Reading

1. Erwin Kreyszig, *Advanced Engineering Mathematics*, 8th Edition, Wiley (1999).

2. G. F. Carrier & C. E. Pearson, *Partial Differential Equations — Theory and Technique*, Academic Press (1988).

3. W. A. Strauss, *Partial Differential Equations: an Introduction*, Wiley (1992).

3.5.4 Calculus in Three Dimensions and Applications — Dr Day — 16 TT

In these lectures, students will be introduced to three-dimensional vector calculus. They will be shown how to evaluate volume, surface and line integrals in three dimensions and how they are related via the divergence theorem and Stokes' theorem. The theory will be applied to problems in physics and optimisation.

Background reading

Erwin Kreyszig, *Advanced Engineering Mathematics* 8th Edition, Wiley. Sections 8.1–8.6, 8.8;
D. E. Bourne & P. C. Kendall, *Vector Analysis and Cartesian Tensors*, Stanley Thornes (1992). Chapters 1–3, Sections 4.1–4.3.

Synopsis

Div, grad and curl in Euclidean coordinates.
(Kreyszig, Sections 8.9–8.11. Bourne & Kendall, Sections 4.4–4.9.)

Volume, surface and line integrals.
Stokes' theorem and the divergence theorem.
Illustration by rederivation of models from Hilary Term, for example, the heat equation from the divergence theorem.
(Kreyszig, Sections 9.1, 9.5–9.9. Bourne & Kendall, Chapters 5, 6.)

Gravity as a conservative force.
Gauss's theorem. The equivalence of Poisson's equation and the inverse-square law.
Informal (geometrical) treatment of Lagrange multipliers.
(Institute Notes to be written by N. M. J. Woodhouse)

Reading

1. Erwin Kreyszig, *Advanced Engineering Mathematics*, 8th Edition, Wiley (1999).
2. D. E. Bourne & P. C. Kendall, *Vector Analysis and Cartesian Tensors*, Stanley Thornes (1992).
3. Institute Notes by N. M. J. Woodhouse
4. Jerrold E. Marsden and Anthony J. Tromba, *Vector Calculus*, McGraw-Hill, Fourth Edition (1996), chapters 7, 8.
5. H. M. Schey, *Div, grad and curl and all that*, (Third Edition) W. W. Norton, (1996).

3.6 Mathematics with Maple

3.6.1 Exploring Mathematics with Maple — Dr Stewart and Dr Lackenby — 16 MT and 16 HT

Background

Mathematicians (like other professionals) use a wide range of generic computer packages: email, word-processors, web-browsers, spreadsheets, database managers and so on. Many, if not most, of the students on the Oxford Mathematics courses will have already used some of these packages; and are encouraged to use the facilities available centrally and in colleges to continue to develop their skills with these during their course.

The use by mathematicians of software developed for handling specific sorts of mathematical problems, especially numerical ones, is well-established; lecture courses in later years will, where appropriate, introduce students to these applications.

Increasingly professional mathematicians use general purpose mathematical packages; sometimes these are called symbolic calculators, or algebraic manipulation packages. Such a package can be used as a super graphics calculator, as a scratchpad, or as a handbook of mathematical functions; its virtue is flexibility. Maple, used in this introductory course, is a good example of such a package.

Aims and Objectives

The aim of the course is to demonstrate the potential of general purpose mathematical packages; to allow students to gain familiarity with one of them (Maple 8); to provide a tool which can be used in the later years of the course.

By the end of Michaelmas term students should be able to

- (i) use the Unix system;
- (ii) edit, save, and use Maple worksheets;
- (iii) manipulate expressions in Maple, and plot simple graphs using Maple;
- (iv) write simple programs in Maple for solving problems in algebra, calculus, and applied mathematics.

By the end of Hilary term students should be able to

- (i) use the **linalg** package within Maple;
- (ii) complete two or three small projects exploring some mathematical problem using Maple;
- (iii) provide in a timely way reports on the projects in the form of commented Maple worksheets.

Synopsis

The Michaelmas term work consists of:

Using the workstation: accounts, passwords, logging in/out; keyboard layout; X-windows environment. Files and directories. Introduction to computer algebra systems: Maple 8; worksheets. Using Maple as a calculator. Manipulation of algebraic formulae. Sets, arrays, tables, and lists. Solution of algebraic equations. Linear Algebra in Maple. Calculus in Maple. Simple graphics. Elementary programming in Maple.

The Hilary term work is based on a menu of mathematical projects; the current list is printed in the second part of the Maple Course Manual.

Access to the system

Undergraduates use the workstations in the Mathematical Institute Computer Laboratory (G17). You must register as a user of the Mathematical Institute network; arrangements will be made to ensure that, as far as possible, you are allocated an account before MT lectures begin.

Students may also access the system through college or individual computers; for details of how to do this they should consult the computing support at their own college. The Maple package may also be installed and used on personally owned computers under the University's site licence; the projects, however, must be submitted through the Maple system on the Mathematical Institute network.

Teaching and Assessment

The course deliberately relies heavily on self-teaching through practical exercises, supported by demonstrations. A manual for the course and a series of worksheets with examples to be worked will be provided.

You will be timetabled for 4 practicals of 2 hours each in Michaelmas term. You will work alone on the projects in Hilary term.

The Moderators in Mathematics are required, when assessing the performance of candidates, to take into account your work on the Maple course. For further information, see the section in the Undergraduate Handbook on examinations.

The logging system of the Mathematics Institute network is used to monitor the work of students on their worksheets.

Reading

Exploring Mathematics with Maple: Students' Guide (Mathematical Institute notes — available from reception).

W. Burkhardt, *First Steps in Maple* (pbk.) (Springer-Verlag, 1994)

F. Wright, *Computing with Maple*, (Chapman & Hall, 2002) (For college reference.)

E. Kreyszig and E. J. Norminton, *Maple Computing Manual for Advanced Engineering Mathematics* (Wiley, 1994) (May be available in libraries.)