MAT syllabus

The quadratic formula. Completing the square. Discriminant. Factorisation. Factor Theorem.

Revision

- The discriminant of a quadratic $ax^2 + bx + c = 0$ is $b^2 - 4ac$. If the discriminant is positive then the quadratic has two real solutions. If the discriminant is zero then there's one (repeated) real solution. If the discriminant is negative then there are no real solutions.
- If $b^2 - 4ac \geqslant 0$, then the solution(s) of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- $ax^2 + bx + c$ can be written as $a(x - \alpha)(x - \beta)$ if $b^2 - 4ac \geqslant 0$, where α and β are roots given by the quadratic formula.
- (Complete the square) We can write $x^2 + bx + c$ in the form $(x + r)^2 + p$ because

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

  This is handy if we're trying to prove that the quadratic is non-negative, because anything squared is non-negative.
- (Difference of two squares) The expression $x^2 - a^2$ factorises as $(x - a)(x + a)$. This comes up quite a lot!
- (Factor Theorem) If $p(a) = 0$ for a polynomial $p(x)$, then $(x - a)$ is a factor of $p(x)$.
- The degree of a polynomial is the highest power of x, so the degree of any quadratic is 2, and the degree of any cubic is 3, for example.
- When sketching the graph of $y = ax^2 + bx + c$, we need to consider whether a is positive or negative (whether it's a "happy" or "sad" quadratic), whether the quadratic has any roots, and where it crosses the y-axis.
- Sometimes a function which is not a quadratic might secretly be a quadratic in a different variable. For example, $y = e^{2x} + e^{x+3} - 1$ is not a quadratic, but if we write $u = e^x$ then we have $y = u^2 + e^3 u - 1$, which is a quadratic. This is sometimes called "changing variable" or "finding the hidden quadratic".

Revision Questions

1. Given the polynomial $p(x) = 2x^3 - 5x^2 + 7x - 3$, evaluate $p(2)$.
2. Find a positive number x which satisfies $x^2 = x + 1$.
3. What is the discriminant of the quadratic $2x^2 + 5x + 1$?
4. For which values of k does $x^2 - x + k = 0$ have exactly two real solutions?
5. For which values of k does $x^4 - x^2 + k = 0$ have exactly two real solutions?
6. How many real solutions does $x^2 + bx + 1 = 0$ have? Find different cases in terms of b.
7. Write $x^2 + 4x + 3$ in the form $(x + a)^2 + b$.
8. Find the maximum or minimum value of the function $f(x) = -2x^2 + 8x + 5$ by completing the square. Is this value a maximum or a minimum?
9. Let $p(x) = x^3 - 13x^2 - 65x - 51$. Check that $p(17) = 0$. Factorise $p(x)$.
10. If $x = 2$ is a zero of the polynomial $p(x)$, what is the corresponding factor of $p(x)$?
11. If $(x - 2)$ is a factor of $p(x)$, what is the value of $p(2)$?
12. Show that $(x - 2)$ is a factor of $f(x) = x^4 - 6x^3 + 13x^2 - 12x + 4$, and factorise $f(x)$.
13. Find all the zeros of the polynomial $p(x) = x^3 - 6x^2 + 11x - 6$.
14. Determine whether $(x - 3)$ is a factor of $p(x) = 2x^4 - 11x^3 + 19x^2 - 20x + 15$.
15. Can a polynomial of degree 3 have exactly two zeros? Why or why not?
16. In each of the following cases, choose a variable u in terms of x to make the function into a quadratic in u. There might be more than one sensible choice of u in each case.
    - $y = 2x^6 + x^3 + 1$,
    - $y = x + \sqrt{2x}$,
    - $y = 3e^{-3x} + 6e^{-6x}$,
    - $y = \dfrac{1 + x}{(1 - x)^2}$.
17. Given that the polynomial $q(x)$ has roots 2, -3, and 1, what could $q(x)$ be? Write down at least two possible polynomials, with different degrees.
18. Given that the polynomial $v(x) = x^3 + 2x^2 + ax + b$ has a double root at $x = 1$, find the values of a and b.

MAT Questions

MAT 2007 Q2

Let

$$f_n(x) = (2 + (-2)^n)\, x^2 + (n + 3)\, x + n^2$$

where n is a positive integer and x is any real number.

(i) Write down $f_3(x)$.

Find the maximum value of $f_3(x)$.

For what values of n does $f_n(x)$ have a maximum value (as x varies)?

[Note you are not being asked to calculate the value of this maximum.]

(ii) Write down $f_1(x)$.

Calculate $f_1(f_1(x))$ and $f_1(f_1(f_1(x)))$.

Find an expression, simplified as much as possible, for

$$f_1(f_1(f_1(\cdots f_1\ (x))))$$

where f_1 is applied k times. [Here k is a positive integer.]

(iii) Write down $f_2(x)$.

The function

$$f_2(f_2(f_2(\cdots f_2(x)))),$$

where f_2 is applied k times, is a polynomial in x. What is the degree of this polynomial?

[See the next page for hints]

Hints

(i) The first part asks us to work out $f_n(x)$ for a small value of n. This is a good trick, whether or not we're prompted by the question to try small values of n. If you've got a guess about the values of n for which $f_n(x)$ has a maximum, you might like to check your guess by testing $n = 1$ and $n = 2$, and perhaps also $n = 4$ and $n = 5$ if you have time.

(ii) Now we're prompted to try $n = 1$. If you didn't check this already, check whether it agrees with your claim in the previous part.

In general, the notation $f(g(x))$ means that we should first work out $g(x)$, and then use that as the input in f, replacing each x in the definition of $f(x)$ with the expression for $g(x)$. Here it's perhaps a bit confusing because f and g are both f_1.

We're asked to extend this to a chain of three f_1s and then a chain of arbitrary length k. Just like the previous part, we're going to need to spot a pattern, and extend to k in general. I find it easier to spot the pattern if I *don't* simplify the coefficients for $f_1(f_1(x))$ and $f_1(f_1(f_1(x)))$ yet; I'll leave the coefficient of x as a product of some 4s, and I won't add together the numbers in the constant coefficient either.

At this stage, don't worry about simplifying your final answer unless you've seen geometric series before (you can leave it as a sum with k terms for now, and come back to this in a few weeks when we've covered that topic).

(iii) Note that we're not being asked to evaluate this polynomial (like we had to in the previous part). Again, it's a good idea to test small values of k, like how in the previous part we worked out $f_1(x)$ and $f_1(f_1(x))$ and $f_1(f_1(f_1(x)))$ before thinking about the general case of a chain of k function evaluations.

Extension

[Just for fun, not part of the MAT question]

- What's the degree of $f_n(f_n(f_n(\cdots f_n(x))))$ where f_n is applied k times, for $n > 2$?
- What's the leading coefficient of that polynomial?
- Let
$$g_n(x) = (8 + (-2)^n)\, x^2 + (n - 3)x + n^2$$
For what values of n does $g_n(x)$ have a maximum value (as x varies)?
- What's the degree of $g_n(g_n(g_n(\cdots g_n(x))))$ where g_n is applied k times?

MAT 2011 Q2 (lightly modified)
Suppose that x satisfies the equation

$$x^3 = 2x + 1.$$

(i) Show that

$$x^4 = x + 2x^2 \qquad \text{and} \qquad x^5 = 2 + 4x + x^2.$$

(ii) For every integer $k \geqslant 0$, we can uniquely write

$$x^k = A_k + B_k x + C_k x^2$$

where A_k, B_k, C_k are integers. So, in part (i), it was shown that

$$A_4 = 0,\ B_4 = 1, C_4 = 2 \qquad \text{and} \qquad A_5 = 2,\ B_5 = 4,\ C_5 = 1.$$

Show that

$$A_{k+1} = C_k, \qquad B_{k+1} = A_k + 2C_k, \qquad C_{k+1} = B_k.$$

(iii) Let

$$D_k = A_k + C_k - B_k.$$

for every integer $k \geqslant 0$. Show that $D_{k+1} = -D_k$ and hence that

$$A_k + C_k = B_k + (-1)^k\,.$$

(iv) Let $F_k = A_{k+1} + C_{k+1}$. Show that

$$F_k + F_{k+1} = F_{k+2}.$$

[See the next page for hints]

Hints

(i) Don't solve for x (solving a cubic equation is not impossible, but it is a bit messy). Instead, think about how the target equation $x^4 = x + 2x^2$ is related to the given equation $x^3 = 1 + 2x$.

For the second equation, we can use a similar idea to turn the x^4 on the left into an x^5. But this time we're not done; on the right-hand side I get $x^2 + 2x^3$. That's not the expression printed in the question... well, the x^2 is correct, but the other term I've got is $2x^3$ and the question has $2 + 4x$. Why are those the same?

(ii) We're being told that we can uniquely write each power of x as a quadratic (we do not need to prove that this is always possible, or prove that the expression is unique). Take a moment to understand how the facts $A_4 = 0$, $A_5 = 2$ relate to the previous part. The printed fact includes a variable k, and we're told that this is true for every integer $k \geqslant 0$. In particular, it's true when $k = 4$, and it's also true when $k = 5$.

Remember how you calculated the values of A_4 and A_5 in the previous part. We're asked to prove something about how A_{k+1} and B_{k+1} and C_{k+1} (on the left of the three equations) relate to A_k and B_k and C_k (on the right of those equations). That's like asking us to calculate the coefficients for x^{k+1} in terms of the expression for x^k. The same two ideas we used in part (i) work here.

(iii) There's a new definition here, for D_k. It's easy to get confused or overwhelmed by a question with lots of notation and variables, but the key here is to understand each new object as it is introduced. We've got D_k on the left, a new and unfamiliar thing, but it's defined to be $A_k + C_k - B_k$, something we know quite a bit about. This is currently the only thing we know about D_k, and we're immediately asked to prove that $D_{k+1} = -D_k$. Our only hope is to "translate" that into a fact about A_k and B_k and C_k and hope that we can use our knowledge of those objects.

"Hence" means that we should use the previous part. We've shown something pretty powerful about D_{k+1}, that it's just (-1) multiplied by D_k. What's D_4 in terms of A_k and B_k and C_k? And what were those values? The final thing we're asked to prove in this part of the question looks similar to D_k; perhaps we could rearrange it to make it clearer what we're being asked to prove about D_k.

(iv) Another new definition, but I can see that there's some relationship between F_{k-1} and A_k and C_k, which we had in the previous parts of this question. So perhaps we can use that to replace all the Fs in the equation here with As and Cs, which we know more about.

Extension

[Just for fun, not part of the MAT question]

- Suppose that x satisfies $x^2 = x + 1$. Prove that $x^3 = 2x + 1$. There are three real numbers which satisfy $x^3 = 2x + 1$. Find all of them. What's this got to do with D_k?