Revision Questions

1. We can rearrange the first equation for x and substitute into the second equation; $x = 1 - 4y$ so $2(1-4y) - y = 3$ which is a linear equation for y with solution $y = -1/9$. Then from the equation $x = 1 - 4y$ we have $x = 13/9$.

2. We can rearrange the second equation for $y = 2-x$ and substitute into the first equation to get $x^2 + 2x + x(2-x) + (2-x)^2 = 5$ which rearranges to $x^2 - 1 = 0$ so $x = 1$ or $x = -1$. We can then use $y = 2 - x$ to find the corresponding values of y. The solution is that (x, y) is $(1, 1)$ or $(-1, 3)$.

3. We can rearrange the first equation for $y = 2 - x^2$ and substitute into the second equation to get $x^4 - 4x^2 + x + 2 = 0$. The first two terms are the difference of two squares, so this is $x^2(x+2)(x-2)+(x+2) = 0$. So $x = 2$ is a solution, or $x^3-2x^2+1 = 0$. This has a root at $x = 1$ and two other roots when $x^2 - x - 1 = 0$ which we can find with the quadratic formula. Substituting these back into the equation $y = 2 - x^2$ we have four solutions for (x, y);

$$(-2, -2) \quad \text{or} \quad (1, 1) \quad \text{or} \quad \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right) \quad \text{or} \quad \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right).$$

4. We can use the binomial theorem here to get $(2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$ which is $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

5. The x^2 term is $\binom{4}{2}(3x)^2(-1)^2$, so the coefficient is 54.

6. $(x + 2)^3 = x^3 + 6x^2 + 12x + 8$, and so the sum of the coefficients is $1 + 6 + 12 + 8 = 27$. Hopefully you spotted that this is 3^3, and that the sum of the coefficients of any polynomial is just the value of that polynomial at $x = 1$. Then the sum of the coefficients of $(x + 2)^{300}$ is 3^{300}.

7. Thinking about the shape of $y = x^2 + 4x + 3$, we should look for any points where $y = 0$ because if there are two roots then the function will be negative in between those roots. We have $x^2 + 4x + 3 = 0$ when $x = -1$ or $x = -3$, so $x^2 + 4x + 3 > 0$ if $x > -1$ or if $x < -3$.

8. a^2 could be as small as zero, because a could be zero. It definitely can't be negative, so that's a lower bound on a^2. On the other hand, a^2 could be almost as large as 4, but not equal to 4 or any larger. So the most that we can say is that $0 \leqslant a^2 < 4$.

9. In the first case, there's nothing we can say about the relationship between ac and bd; those could be any two numbers.

   In the second case, we can say something! We have $c < d$ and then because $a > 0$ we can multiply each side by a to get $ac < ad$. Separately, we can start with $a < b$ and multiply by d to get $ad < bd$ since $d > 0$ (because $d > c$ and $c > 0$). Combining these two results, we have $ac < bd$.

10. There are $2^5 = 32$ possible sequences of heads/tails that I could get when I flip five coins. Exactly three of the coins are heads (and two are tails) in some of these sequences; if I think about choosing different combinations of coins to be the ones that come up heads then I can see that the number of sequences with exactly three heads is precisely $\binom{5}{3} = 10$. So the probability is $\frac{10}{32} = \frac{5}{16}$.

11. We could consider what happens as we deal out the cards one at a time. It doesn't matter if the first card is even or odd, but then the probability the second card is the opposite parity is $\frac{3}{5}$. If that continues the pattern, then we've got one odd number and one even number dealt out and four cards to go. The probability that the next one continues the pattern is $\frac{2}{4}$, then if we continue to think about the probability that each card fits the pattern we have the probabilities $\frac{2}{3}$ then $\frac{1}{2}$ then $\frac{1}{1}$. Multiplying all these probabilities together, because we want all these events to happen one after the other, we get a probability of $\frac{1}{10}$.

    Alternatively, consider trying to count how many of the $6! = 720$ possible shuffles have the alternating property. There are some that start with an even number and some that start with an odd number. Consider the ones that start with an even number; we must have the 2 and 4 and 6 cards as the first and third and fifth cards dealt, and there are $3! = 6$ ways to do this. Similarly, there are 6 permutations for the odd cards, so $6 \times 6 = 36$ shuffles match this pattern. Separately, there are 36 shuffles that start with an odd number and then alternate, for a total of 72 out of a possible 720. That's a probability of $\frac{72}{720} = \frac{1}{10}$.

12. There are $\binom{10}{3}$ ways to choose the finalists. If contestant 1 is chosen, there are $\binom{8}{2}$ ways to choose two more finalists who are not contestant 2. The probability is

$$\frac{\binom{8}{2}}{\binom{10}{3}} = \frac{7}{30}$$

13. There are $\binom{20}{4}$ ways for the teacher to select four students. How many of those possibilities have exactly three girls and one boy? There are $\binom{10}{3}$ ways to choose the girls for such a group, and in each of those cases there are $\binom{10}{1}$ ways to choose the boy for the group. The probability is therefore

$$\frac{\binom{10}{3}\binom{10}{1}}{\binom{20}{4}} = \frac{80}{323}$$

14. The probability of getting all ten questions right is $\left(\frac{1}{5}\right)^{10}$. The probability of getting exactly 9 right is $\binom{10}{9}\left(\frac{1}{5}\right)^9\left(\frac{4}{5}\right)$. The probability of getting exactly 8 right is $\binom{10}{8}\left(\frac{1}{5}\right)^8\left(\frac{4}{5}\right)^2$. Add those together!

MAT Questions

MAT 2009 Q4

(i) First let's find the gradient of the tangent to the curve at $x = a$. The gradient of $y = x^2$ is $2x$, which is $2a$ at $x = a$. So the gradient of the normal at that point is $\dfrac{-1}{2a}$, provided that $a \neq 0$. The line L also goes through the point (a, a^2). So the equation is $y = \dfrac{-1}{2a}(x - a) + a^2$. There's a special case if $a = 0$, when the normal is the vertical line $x = 0$.

(ii) This goes through P if

$$1 = \frac{-1}{2a}(0 - a) + a^2.$$

This rearranges to $2a^2 = 1$ with solutions $a = \pm\sqrt{\frac{1}{2}}$. The special case $a = 0$ also gives a line that goes through the point P.

(iii) Using Pythagoras, $|QP|^2 = (a - 0)^2 + (a^2 - 1)^2$ so $|QP| = \sqrt{a^2 + (a^2 - 1)^2}$.

(iv) Let's find the minimum value of the expression $a^2+(a^2-1)^2$. That expands to a^4-a^2+1. The derivative (with respect to a) is $4a^3 - 2a$ which is zero for $a = 0$ or for $a = \pm\sqrt{\frac{1}{2}}$. We should check the value; the expression is 1 at $a = 0$, and it's $\frac{3}{4}$ for $a = \pm\sqrt{\frac{1}{2}}$, so the minimum happens when $a = \pm\sqrt{\frac{1}{2}}$.

The actual value of $|QP|$ is the square root of the expression we considered here, but $\sqrt{x}$ is an increasing function, so the minimum occurs at the same value of a.

(v) We could adapt the algebra we've just done, to put R somewhere else on the y-axis, at $(0, r)$ say. Then the squared distance becomes $a^4 + (1 - 2r)a^2 + r^2$. The derivative is $4a^3 + 2(1 - 2r)a$, which is zero if $a = 0$ or if $4a^2 + (1 - 2r) = 0$. I can see that if $(1 - 2r) > 0$ then this second case has no solutions. So if I take R to be $(0, 0.3)$ for example, then the distance $|RQ|$ is smallest for the unique value of $a = 0$.

Alternatively, put the point R anywhere below the curve $y = x^2$. If you imagine expanding a circle centered on such a point R, it'll eventually meet the parabola at a single point (because the circle curves in the opposite direction to the parabola). This point corresponds to a unique value of a which minimises the distance $|RQ|$.

There are other points R that work (above the parabola, but "close" to the parabola).

Extension

- The shortest distance from a point to a curve occurs when a line through that point is normal to the curve. This is like "dropping a perpendicular" to the curve.

MAT 2011 Q3

(i) The gradient of the line is m. The gradient of the cubic is $3x^2 - 1$, which is $3b^2 - 1$ at $x = b$. The line and the curve tough at $x = b$, so they have the same gradient at that point.

(ii) I haven't used the fact that the line goes through the point B yet. The value of the cubic is $b^3 - b$, and the value of the line is $m(b - a)$. This rearranges to

$$a = \frac{mb - b^3 + b}{m}.$$

Next I'll substitute to get rid of the m, using $m = 3b^2 - 1$, and then the numerator simplifies a bit to give the expression in the question.

(iii) Note that the cubic on the left-hand side has zero x^2 coefficient. If we multiply out the right-hand side, we get $(x^2 - 2bx + b^2)(x - c) = x^3 - 2bx^2 - cx^2 + b^2x + 2bcx - b^2c$. Comparing the coefficient of x^2, we conclude that $-c - 2b = 0$ so $c = -2b$.

(iv) The expression we just found is really large if b is very negative, or if b is close to $\pm 1/\sqrt{3}$. If b is very negative, then the approximate value of a is -1.5×10^6, but we know that $a < b$. We're also told that $b < 0$ so b is close to $-1/\sqrt{3}$.

(v) The area is

$$\int_b^c m(x - a) - x^3 + x \,\mathrm{d}x$$

where I've written it as the area underneath the line minus the area under the cubic (or you can think of this as an expression for the difference in values between the curves). I'll use the fact from part (iv), the expansion we worked out in part (iv), and the fact that $c = -2b$ to write this as

$$-\int_b^{-2b} x^3 - 3b^2x + 2b^3 \,\mathrm{d}x = -\left[\frac{x^4}{4} - 3b^2\frac{x^2}{2} + 2b^3x\right]_b^{-2b} = \frac{27}{4}b^4$$

If we remember that b is negative, we can make this expression as large as possible by making b as negative as possible. The corresponding value of a (using the equation in part (ii)) is -1. The corresponding value of R is $27/4$.

Extension

- The cubic on the left is the difference between the cubic and the line, so it has roots when those curves meet. The root at b is a repeated root because the line and the cubic *touch* there. There are no other roots, and the leading coefficient is 1.
- This is the same cubic, but transformed by a stretch factor $k^{1/2}$ parallel to the x-axis and by a stretch factor of $k^{3/2}$ parallel to the y-axis. The maximum area for R becomes $27k^2/4$.