Revision Questions

1. The derivative of x^a is ax^{a-1} so the derivative of this expression is $17x^{16} + 17x^{-18}$.

2. Remember that $\sqrt{x} = x^{1/2}$ and $\sqrt[3]{x} = x^{1/3}$, so the derivative of this expression is $x^{-1/2} + x^{-2/3}$, which we could write as $\frac{1}{\sqrt{x}} + \frac{1}{x^{2/3}}$ if we wanted to.

3. Remember that the derivative of a constant is 0, so the derivative of this expression is just $-3e^{3x}$.

4. We need to find the value of the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 2$ because that's equal to the gradient of the tangent. We can differentiate to find $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x + 2x$ so that gradient we wanted is $e^2 + 4$. We also want the tangent to have the same value at $x = 2$ as the curve; that's $e^x + x^2$ at $x = 2$, which is also $e^2 + 4$. So our tangent is $y = (e^2 + 4)(x - 1)$.

5. First find the derivative at $x = 3$, which is 6 for this parabola. That's the gradient of the tangent, and the normal is at right-angles to the tangent, so it has gradient $-\frac{1}{6}$. We have $y = -\frac{x}{6} + c$ and we want the normal to go through the point $(3, 9)$. So we want $c = \frac{19}{2}$.

6. The turning points must have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ so we must have $4x^3 - 6x^2 + 2x = 0$. That happens when $x = 0$ or when $2x^2 - 3x + 1 = 0$ which happens when $(2x - 1)(x - 1) = 0$, which is either $x = 1$ or $x = \frac{1}{2}$.

   Now find the second derivative to check whether these are minima or maxima. We have $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 12x + 2$, which is positive for $x = 0$, negative for $x = \frac{1}{2}$, and positive for $x = 1$. So we have a (local) minimum, then a (local) maximum, then a (local) minimum.

   The function is decreasing for $x < 0$, then increasing for $0 < x < \frac{1}{2}$, then decreasing for $\frac{1}{2} < x < 1$ then increasing for $x > 1$.

7. The line definitely goes through A, which doesn't move. The thing we learn from "differentiation from first principles" is that the gradient of the line gets closer and closer to the derivative of the function at A.

   The derivative is $3x^2 + 2x + 1$ which is 6 at $x = 1$. The value is 4, so the tangent is $y = 6x - 2$. So if the line through A and B is $y = mx + c$ then m gets closer and closer to 6 and c gets closer and closer to -2.

8. First find the points where $y = 0$. We have $(x+3)(x+1) = 0$ so these points are at $x = -1$ or $x = -3$. In between, we have $y < 0$ (by considering the graph).

   So we want $-\int_{-3}^{-1} x^2+4x+3\,\mathrm{d}x$. That minus sign out the front is because the function is negative in this region. This works out to be $\frac{4}{3}$.

9. - $\int \frac{x+3}{x^3}\,\mathrm{d}x = \int \frac{1}{x^2} + \frac{3}{x^3},\mathrm{d}x = -\frac{1}{x} - \frac{3}{2x^2} + c$ where c is a constant.
   - $\int \sqrt[3]{x}\,\mathrm{d}x = \int x^{1/3}\,\mathrm{d}x = \frac{3}{4}x^{4/3} + c$ where c is a constant.
   - $\int \left(\left(x^2\right)^3\right)^5\,\mathrm{d}x = \int x^{30}\,\mathrm{d}x = \frac{x^{31}}{31} + c$ where c is a constant.
   - $\int \left(x^2+1\right)^3\,\mathrm{d}x = \int x^6 + 3x^4 + 3x^2 + 1\,\mathrm{d}x = \frac{x^7}{7} + \frac{3x^5}{5} + x^3 + x + c$ where c is a constant.

10. The graph of $f(-x)$ is the graph of $f(x)$ reflected in y-axis. Also, note that if we reflect the interval $-1 \leqslant x \leqslant 1$ in the y-axis then we get the same interval back. On the left-hand side, we're finding the area under $f(x)$ (or maybe negative the area in any regions where f is negative). On the right-hand side, we're calculating exactly the same area, but with the shape of the graph reflected.

11. First consider the graph $y = \frac{1}{x}$. The area under the graph between $x = 1$ and $x = 10$ is I_1. Now consider stretching that region by a factor of 10 parallel to the x-axis, and squashing it by a factor of 10 parallel to the y-axis. The area will be the same, and (amazingly!) any point that was on the curve $y = \frac{1}{x}$ is still on the graph after these transformations. So I_2, the area under the graph between 10 and 100 is equal to I_1.

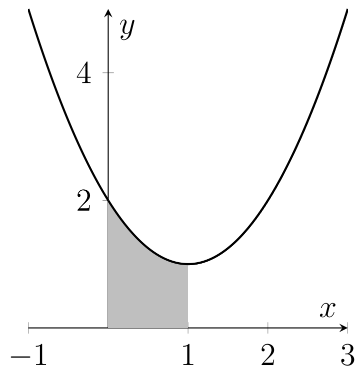
    This means that
    $$\int_1^{100} \frac{1}{x}\,\mathrm{d}x = \int_1^{10} \frac{1}{x}\,\mathrm{d}x + \int_{10}^{100} \frac{1}{x}\,\mathrm{d}x = I_1 + I_2 = 2I_1.$$
    But similarly, if we think about stretching the graph again in the same way, we find that $\int_{100}^{1000} \frac{1}{x}\,\mathrm{d}x$ is also equal to I_1. By setting N to be a large power of ten, we can make $\int_1^N \frac{1}{x}\,\mathrm{d}x$ arbitrarily large.

12. Note that $\frac{x^2}{1+x^2} + \frac{1}{1+x^2} = 1$ so $I_3 + I_4 = \int_1^3 1\,\mathrm{d}x = 2$. So $I_3 + I_4 = 2$.

13. Note that $\frac{x^4}{1+x^2} = x^2 - \frac{x^2}{1+x^2}$ so this new integral is $\int_1^3 x^2\,\mathrm{d}x - I_4 = 8\frac{2}{3} - I_4$.

MAT 2007 Q3

(i) This quadratic has a turning point at $(1, 1)$ and a positive coefficient of x^2. The y-intercept is 2. Here's my sketch;



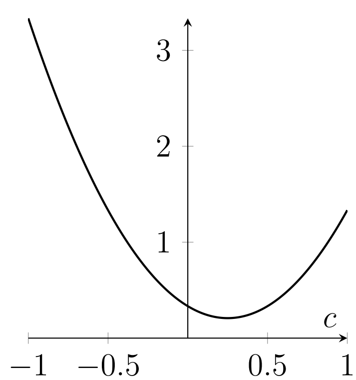
(ii) $(x-c)^2$ is positive or zero, and c^2 is positive or zero, so $(x-c)^2+c^2$ is positive or zero. The integral of a function $f(x) \geqslant 0$ is positive or zero, because the integral is the area under the curve.

(iii) We have

$$I(c) = \int_0^1 x^2 - 2cx + 2c^2 \,\mathrm{d}x = \frac{1}{3} - c + 2c^2$$

(iv) Differentiate this quadratic with respect to c; the minimum occurs when $4c - 1 = 0$, which is when $c = \frac{1}{4}$. The value of $I\left(\frac{1}{4}\right)$ is $\frac{5}{24}$.

(v) Remember that $-1 \leqslant \sin\theta < 1$. Let's sketch the quadratic $2c^2 - c + \frac{1}{3}$ against c for $-1 \leqslant c \leqslant 1$



This quadratic has a minimum in this interval, so we check the ends of the interval for a maximum. $I(1) = \frac{4}{3}$ and $I(-1) = \frac{10}{3}$. The maximum value of $I(\sin\theta)$ is therefore $\frac{10}{3}$.

Extension

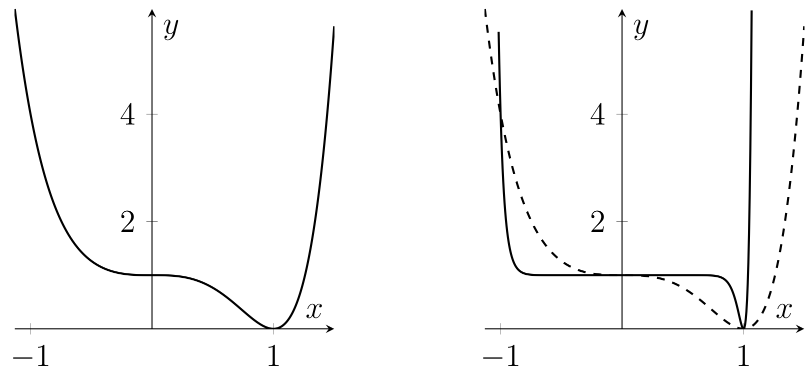
- This is a translation of $x^6 + x^2$. That function has a minimum at $x = 0$ and increases away from $x = 0$. We can make the integral from 0 to 1 small by choosing $c = 1/2$.

- First suppose that $a \geqslant 0$. In this case, the quadratic has a minimum turning point (or no turning point if $a = 0$), so we compare the values at -1 and 1. These are $a - b + c$ and $a + b + c$. So there are two sub-cases; if $b \geqslant 0$ then the maximum value is $a + b + c$, but if $b < 0$ then the maximum value is $a - b + c$.

  However, if $a < 0$ then the maximum might occur at the turning point where $2ax + b = 0$, provided that $-1 \leqslant -b/2a \leqslant 1$. We should check the value at this turning point; it's $c - \frac{b^2}{4a}$. Depending on whether this is larger or smaller than $a - b + c$ or $a + b + c$, this could be the maximum value.

MAT 2009 Q3

(i) The function inside the brackets is $x^3 - 1$. That's 0 at $x = 1$ and it's -1 at $x = 0$. It's negative for $x < 1$. Here's my sketch of the square of that function, on the left below.



(ii) For higher powers of n, the function x^{2n-1} is approximately zero for $|x| < 1$, and it grows rapidly outside that range. So $x^{2n-1} - 1$ is about -1 for the range $|x| < 1$, but it shoots up to high positive values soon after $x = 1$ and it shoots down to very negative values just before $x = -1$. If we square that function, we'll get something that's about 1 for most of the range $|x| < 1$, but near the edges of that region two strange things will happen. Near $x = -1$, the function inside the brackets just gets really negative. For x near 1, the function inside the brackets increases to zero then increases to high positive values. For the square of the function, this is a decrease to zero first, then an increase to high positive values. See my sketch above, on the right, with the dashed line indicating my previous sketch.

(iii) We have

$$\int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 x^{4n-2} - 2x^{2n-1} + 1\,\mathrm{d}x = \left[\frac{x^{4n-1}}{4n-1} - \frac{x^{2n}}{n} + x\right]_0^1 = \frac{1}{4n-1} - \frac{1}{n} + 1$$

where the contributions from the lower limit $x = 0$ are all zero because those powers of x give zero for $n \geqslant 1$ a whole number.

(iv) We would like

$$1 + \frac{1}{4n-1} - \frac{1}{n} \leqslant 1 - \frac{A}{n+B}$$

for all $n \geqslant 1$. This rearranges (being careful not to multiply by negative numbers) to the inequality

$$0 \geqslant (4A-3)n^2 + (1-A-3B)n + B$$

If the coefficient of n^2 is positive, this clearly doesn't work (because the right-hand side will get really large and positive for large enough n), so we must have $A \leqslant 3/4$.

(v) If the coefficient of n^2 is zero, then $A = 3/4$. We still need

$$0 \geqslant \left(\frac{1}{4} - 3B\right) n + B$$

for all $n \geqslant 1$. For this to work, the linear function on the right-hand side must have negative or zero gradient, and the value at $n = 1$ must be negative or zero. We therefore require both $B \geqslant 1/12$ and also $B \geqslant 1/8$. Since we need both of these to hold, we must have $B \geqslant 1/8$.

Quick check that if $A = 3/4$ and $B = 1/8$ then the inequality is in fact true.

Extension

- If $n = 1/2$ then the function is constant and zero. The integral is zero.
- We need $n > 1/4$. If $n \leqslant 1/4$ then the integral doesn't exist, because x^{4n-1} gets very large near $x = 0$.
- Sketch: