

## MAT syllabus

Derivative of  $x^a$ , including for fractional exponents. Derivative of  $e^{kx}$ . Derivative of a sum of functions. Tangents and normals to graphs. Turning points. Second order derivatives. Maxima and minima. Increasing and decreasing functions. Differentiation from first principles. Indefinite integration as the reverse of differentiation. Definite integrals and the signed areas they represent. Integration of  $x^a$  (where  $a \neq -1$ ) and sums thereof.

## Revision

- The derivative of  $x^a$  is  $ax^{a-1}$ , including for fractional exponents like  $a = \frac{1}{2}$ .
- If  $k$  is a constant then the derivative of  $e^{kx}$  is  $ke^{kx}$ .
- If  $a$  is a constant then the derivative of  $af(x)$  is  $a$  times the derivative of  $f(x)$ .
- The derivative of  $y_1 + y_2$  is (the derivative of  $y_1$ ) + (the derivative of  $y_2$ ). Perhaps this looks too obvious to need stating, but remember that, for example, the square of  $y_1 + y_2$  is not equal to (the square of  $y_1$ ) + (the square of  $y_2$ ).
- The tangent to a graph at a particular point is a line which has the same value and derivative as the graph at that point. So if we want the tangent to the graph  $y = x^2$  at  $x = 3$ , we need the value of  $y$  (which is 9), and the value of the derivative (which is 6). The derivative of a line is its gradient, so we can write  $y = 6x + c$  and solve for  $c$  using the value at  $x = 3$  to get  $y = 6x - 9$ .
- The normal to a graph is a line which has the same value and is at right angles to the tangent. Two lines are at right angles if their gradients multiply to  $-1$ . So at the point above, we would want  $y = -\frac{1}{6}x + c$  and, since the line goes through  $(3, 9)$ , we have  $c = \frac{19}{2}$ .
- If the derivative changes sign (+/−) at a point, that’s a turning point. You’ll have zero derivative at the turning point, but that’s not actually sufficient for the derivative to *change* sign (e.g.  $x^3$  has zero derivative at  $x = 0$ , but that’s not a turning point because the derivative is positive on both sides). A point with zero derivative is called a stationary point.
- The derivative of a derivative is called the second derivative. You can work out the derivatives one at a time. So the second derivative of  $x^a$  would be the derivative of  $ax^{a-1}$ , which is  $a(a-1)x^{a-2}$ . The second derivative of  $e^{kx}$  is  $k^2e^{kx}$ . The second derivative is the rate of change of the derivative.
- “Maxima” is the plural of “maximum”. “Minima” is the plural of “minimum”. A turning point is a local maximum if the second derivative is negative at that point,

or it's a local minimum if the second derivative is positive. The word "local" here means that very near to that point, the function takes its maximum value. Overall, the function might have several local maxima, or none, and it might increase without bound (like  $y = x$  for example) so just having derivative zero might not mean that that's biggest value of the function. Over an interval like  $[0, 1]$  the function might take its maximum at a local maximum, or maybe at one of the endpoints (like how  $y = x$  would take its maximum value at  $x = 1$  over that interval).

- If the derivative is positive, that's an increasing function. If it's negative, that's a decreasing function. In general a function might increase in some regions and decrease in other regions.
- If you have two points on a graph, you can join the line between them – that's called the chord. If you move the second point closer and closer to the first point, then the gradient of the chord gets closer and closer to the gradient of the tangent, which is the value of the derivative at that point. Calculating the gradient of the chord is a nice and sensible thing to do; it's just  $\frac{y_2 - y_1}{x_2 - x_1}$ , so this is called a "first principles" approach to differentiation.
- Indefinite integration (without limits as in  $\int x^2 dx$ ) is the reverse of differentiation in the sense that if the derivative of  $f(x)$  is  $g(x)$  then the indefinite integral of  $g(x)$  is  $f(x) + c$  where  $c$  could be any constant. You can use this to integrate any function which you could have got as the result of some differentiation.
- The integral of  $x^n$  is  $\frac{x^{n+1}}{n+1}$ , provided that  $n \neq -1$ .
- A definite integral (with limits as in  $\int_1^2 x^2 dx$ ) is written like  $\int_a^b f(x) dx$  where  $a$  and  $b$  are the two end-points. This is the difference in value of the indefinite integral at the two end-points;  $F(b) - F(a)$  where the derivative of  $F(x)$  is  $f(x)$ .
- If  $f(x) > 0$  for  $a < x < b$  then  $\int_a^b f(x) dx$  is the area of the region bounded by the curve  $y = f(x)$ , the  $x$ -axis, and the lines  $x = a$  and  $x = b$ .
- If  $f(x) < 0$  for  $a < x < b$  then  $\int_a^b f(x) dx$  is minus one times the area of the region bounded by the curve  $y = f(x)$ , the  $x$ -axis, and the lines  $x = a$  and  $x = b$ . Areas are supposed to be positive. The integral here is sometimes called the "signed area" to reflect the fact that it's got a minus sign.
- If  $f(x)$  is sometimes positive and sometimes negative in  $a < x < b$  then we can split into separate regions where  $f(x)$  is positive or negative before applying the above.
- $\int_a^b f(x) dx = -\int_b^a f(x) dx$
- $\int_a^\infty f(x) dx$  means the limit of  $F(b) - F(a)$  for very large  $b$  (if this limit exists!). Formal knowledge of limits is not expected.

## Revision Questions

1. Differentiate  $x^{17} - x^{-17}$  with respect to  $x$ .
2. Differentiate  $2\sqrt{x} + 3\sqrt[3]{x}$  with respect to  $x$ .
3. Differentiate  $1 - e^{3x}$  with respect to  $x$ .
4. Find the tangent to the curve  $y = e^x + x^2$  at  $x = 2$ .
5. Find the normal to the parabola  $y = x^2$  at  $x = 3$ .
6. Find the turning points of the curve  $y = x^4 - 2x^3 + x^2$ . Identify whether the turning points are maxima or minima. For which values of  $x$  is  $y = x^4 - 2x^3 + x^2$  increasing? For which values of  $x$  is it decreasing?
7. Two points  $A$  and  $B$  are on the curve  $y = x^3 + x^2 + x + 1$ .  $A$  is fixed at  $(1, 4)$ . The point  $B$  moves along the curve towards  $A$ . What happens to the line through  $A$  and  $B$ ?
8. Find the area enclosed between the polynomial  $y = x^2 + 4x + 3$  and the  $x$ -axis.

9. Find

$$\int \frac{x+3}{x^3} dx, \quad \int \sqrt[3]{x} dx, \quad \int \left( (x^2)^3 \right)^5 dx, \quad \int (x^2 + 1)^3 dx$$

10. By thinking about the area that the integral represents, explain why

$$\int_{-1}^1 f(x) dx = \int_{-1}^1 f(-x) dx.$$

11. Let  $I_1 = \int_1^{10} \frac{1}{x} dx$  and let  $I_2 = \int_{10}^{100} \frac{1}{x} dx$  (you are not expected to calculate either of these integrals). By considering a rescaling of the graph  $y = \frac{1}{x}$ , and the area under that graph, prove that  $I_1 = I_2$ . Deduce that  $\int_1^N \frac{1}{x} dx$  with  $N > 1$  can be made arbitrarily large by increasing  $N$ .

12. Let  $I_3 = \int_1^3 \frac{1}{1+x^2} dx$ . Let  $I_4 = \int_1^3 \frac{x^2}{1+x^2} dx$ . Without calculating either integral, write down a relationship between  $I_3$  and  $I_4$ .

13. Calculate  $\int_1^3 \frac{x^4}{1+x^2} dx$  in terms of  $I_3$  and/or  $I_4$  from the previous question.

## MAT questions

### MAT 2007 Q3

Let

$$I(c) = \int_0^1 ((x - c)^2 + c^2) \, dx$$

where  $c$  is a real number.

- (i) Sketch  $y = (x - 1)^2 + 1$  for the values  $-1 \leq x \leq 3$  and show on your graph the area represented by the integral  $I(1)$ .
- (ii) Without explicitly calculating  $I(c)$ , explain why  $I(c) \geq 0$  for any value of  $c$ .
- (iii) Calculate  $I(c)$ .
- (iv) What is the minimum value of  $I(c)$  (as  $c$  varies)?
- (v) What is the maximum value of  $I(\sin \theta)$  as  $\theta$  varies?

[\[See the next page for hints\]](#)

### Hints

- (i) That's a quadratic, and someone has already completed the square for us. Where's the turning point?
- (ii) If in doubt, try sketching  $(x - c)^2 + c^2$ . What's the value of that function when  $x = 0$ ? What about when  $x = 1$ ? Where's the turning point?
- (iii) Time to integrate a quadratic. It's probably easiest to multiply it out and then integrate term-by-term.
- (iv) Because the independent variable here is  $c$ , we might think about differentiating with respect to  $c$ . If that's not something you're happy with, try writing out your expression  $I(c)$  from the previous part as  $I(x)$  instead. The reason the question doesn't use  $x$  here is because  $x$  is already being used as the dummy variable in the integral above.
- (v) Be careful, we're now being asked for the maximum value, not the minimum value. Remember that  $\sin \theta$  can only take values between  $-1$  and  $1$ .

### Extension

[Just for fun, not part of the MAT question]

- Let

$$I(c) = \int_0^1 ((x - c)^6 + (x - c)^2) dx$$

Without detailed calculation, find the value of  $c$  where  $I(c)$  attains its minimum value.

- Consider the quadratic  $f(x) = ax^2 + bx + c$ . Depending on the values of  $a$ ,  $b$ , and  $c$ , there are several possibilities for the maximum value of  $f(\sin \theta)$ . Describe the different cases, and the value of  $f(\sin \theta)$  in each case.

**MAT 2009 Q3**

For a positive whole number  $n$ , the function  $f_n(x)$  is defined by

$$f_n(x) = (x^{2n-1} - 1)^2.$$

- (i) Sketch the graph of  $y = f_2(x)$  labelling where the graph meets the axes.
- (ii) On the same axes sketch the graph of  $y = f_n(x)$  where  $n$  is a large positive integer.
- (iii) Determine

$$\int_0^1 f_n(x) \, dx.$$

- (iv) The *positive* constants  $A$  and  $B$  are such that

$$\int_0^1 f_n(x) \, dx \leq 1 - \frac{A}{n+B} \quad \text{for all } n \geq 1.$$

Show that

$$(3n-1)(n+B) \geq A(4n-1)n,$$

and explain why  $A \leq 3/4$ .

- (v) When  $A = 3/4$ , what is the smallest possible value of  $B$ ?

[\[See the next page for hints\]](#)

### Hints

- (i) We have  $f_2(x) = (x^3 - 1)^2$ . Where are the intercepts? What happens when  $x$  is very large?
- (ii) First, think about  $x^{2n-1}$  for large  $n$ , in different cases depending on whether  $x$  is large, small, positive, negative, zero.
- (iii) Just like we did for the other MAT question, it's probably best to multiply out the square here and integrate term-by-term.
- (iv) You've got an expression for the left-hand side. This is supposed to be true for all  $n$ . To get your head around that, imagine plugging in different values of  $n$ , including some when  $n$  is small / large.
- (v) Plug in  $A = 3/4$  and expand both sides of the inequality. Rearrange to get everything on one side. Could this inequality really be true for all positive whole numbers  $n$ ? How?

### Extension

[Just for fun, not part of the MAT question]

- What happens if, instead of being a positive whole number,  $n = \frac{1}{2}$ ?
- The expression that you've found for  $\int_0^1 f(x) dx$  holds for any real number  $n$ , provided that  $n > n_{\max}$  where  $c$  is some real number. Find the largest real value of  $n$  for which your expression does not hold, and explain why it doesn't hold for any smaller value of  $n$ .
- Sketch

$$y = 1 - \frac{3x - 1}{x(4x - 1)} \quad \text{for } x > n_{\max}$$