MAT syllabus

Laws of logarithms and exponentials. Solution of the equation $a^x = b$.

Revision

- $a^m a^n = a^{m+n}$ for any positive real number a and any real numbers m and n.
- $(a^m)^n = a^{mn}$ for any positive real number a and any real numbers m and n.
- $a^{-n} = \dfrac{1}{a^n}$ for any positive real number a and any real number n.
- $a^0 = 1$ for any non-zero real number a.
- The solution to $a^x = b$ where a and b are positive numbers (with $a \neq 1$) is $\log_a(b)$. In this expression, the number a is called the base of the logarithm.
- $\log_a(x)$ is a function of x which is defined when $x > 0$. Like with $\sin x$, sometimes the brackets are omitted if it's clear what the function is being applied to, so we might write $\log_a x$.
- $\log_a x$ doesn't repeat any values; if $\log_a x = \log_a y$ then $x = y$.
- Note the special case $\log_a a = 1$ because $\log_a a$ is the solution x to the equation $a^x = a$, and that solution is 1.
- In fact, $\log_a(a^x) = x$.
- In that sense, the logarithm function is the inverse function for $y = a^x$.
- $a^{\log_a x} = x$.
- $\log_a(xy) = \log_a(x) + \log_a(y)$.
- $\log_a(x^k) = k \log_a x$ including $\log_a \dfrac{1}{x} = -\log_a x$.
- There's a mathematical constant called e, which is just a number (it's about 2.7).
- e^x is called the exponential function.
- The laws of indices and laws of logarithms above hold when the base a is equal to e.
- $\log_e x$ is sometimes written as $\ln x$ and the function is sometimes called the natural logarithm.

Revision Questions

1. Simplify $(2^3)^4$ and $(2^4)^3$ and $2^4 2^3$ and $2^3 2^4$.
2. Solve $x^{-2} + 4x^{-1} + 3 = 0$.
3. Simplify $\log_{10} 3 + \log_{10} 4$ into a single term.
4. Write $\log_3(x^2 + 3x + 2)$ as the sum of two terms, each involving a logarithm.
5. Solve $\log_x(x^2) = x^3$.
6. Solve $\log_x(2x) = 3$ for $x > 0$.
7. Solve $\log_{x+5}(6x + 22) = 2$.
8. Let $a = \ln 2$ and $b = \ln 5$, and write the following in terms of a and b.

$$\ln 1024, \quad \ln 40, \quad \ln \sqrt{\frac{2}{5}}, \quad \ln \frac{1}{10}, \quad \ln 1.024.$$

9. Expand $(e^x + e^{-x})(e^y - e^{-y}) + (e^x - e^{-x})(e^y + e^{-y})$.
   Expand $(e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})$.
10. Solve $2^x = 3$. Solve $0.5^x = 3$. Solve $4^x = 3$.
11. For which values of x (if any) does $1^x = 1$? For which values of x (if any) does $1^x = 3$?
12. For what values of b (if any) does $0^b = 0$? For what values of a (if any) does $a^0 = 0$?
13. Given $\log_{10}(\log_{10} x) = 6$, how many zeros are there at the end of the number x?
14. Solve $e^x + e^{-x} = 4$.
    How many solutions are there to $e^x + e^{-x} = c$? Identify different cases in terms of c.
15. Prove that $\ln(N + \sqrt{N^2 - 1}) = -\ln(N - \sqrt{N^2 - 1})$ for any number $N \geqslant 1$.
16. Consider the equation $x^y = y^x$ with $x, y > 0$. Use logarithms to turn this into an equation of the form $f(x) = f(y)$. [Harder] Sketch $f(x)$.
17. Simplify $a^{k \log_a b}$ for positive numbers a, b, k.
18. Consider the number $x = \log_a b \log_b c$. By simplifying a^x, show that $x = \log_a c$.
19. Similarly, show that $\log_a b = \dfrac{\log_c b}{\log_c a}$ for positive numbers a, b, c, and hence $\log_a b = \dfrac{\ln b}{\ln a}$.

MAT questions

MAT 2007 Q1I
Given that a and b are positive and

$$4\left(\log_{10} a\right)^2 + \left(\log_{10} b\right)^2 = 1,$$

then the greatest possible value of a is

(a) $\dfrac{1}{10}$, (b) 1, (c) $\sqrt{10}$, (d) $10^{\sqrt{2}}$.

MAT 2008 Q1B
Which is the smallest of these values?

(a) $\log_{10} \pi$, (b) $\sqrt{\log_{10}\left(\pi^2\right)}$, (c) $\left(\dfrac{1}{\log_{10} \pi}\right)^3$, (d) $\dfrac{1}{\log_{10} \sqrt{\pi}}$.

MAT 2008 Q1E
The highest power of x in

$$\left\{\left[\left(2x^6+7\right)^3 + \left(3x^8-12\right)^4\right]^5 + \left[\left(3x^5-12x^2\right)^5 + \left(x^7+6\right)^4\right]^6\right\}^3$$

is

(a) x^{424}, (b) x^{450}, (c) x^{500}, (d) x^{504}.

MAT 2010 Q1E
Which is the largest of the following four numbers?

(a) $\log_2 3$, (b) $\log_4 8$, (c) $\log_3 2$, (d) $\log_5 10$.

MAT 2012 Q1C (modified)
Which is the *smallest* of the following numbers?

(a) $\left(\sqrt{3}\right)^3$, (b) $\log_3\left(9^2\right)$, (c) $\left(3\sin 60^\circ\right)^2$, (d) $\log_2\left(\log_2\left(8^5\right)\right)$.

[See the next page for hints]

Hints

MAT 2007 Q1I

- If I squint at the left-hand side, it looks a bit like the sum of two squares. Let's write $x = \log_{10} a$ and $y = \log_{10} b$ and see what happens.

MAT 2008 Q1B

- When is x bigger than $\sqrt{2x}$? When is x bigger than $2/x$?

  You'll need to use the fact that $1 < \pi < 10$, but you shouldn't need to use any more detailed knowledge of the value of π than that.

MAT 2008 Q1E

- What's the highest power of x in $(2x^6 + 7)^3$? Do not multiply out! Now look at the other terms too.

  We can ignore the outer-most power of 3 while we're comparing terms, but don't forget about it at the end.

MAT 2010 Q1E

- You can evaluate one of these exactly. Which one? Next, I would aim to compare the others to that one.

  Here's a strategy to do that sort of comparison; let's say that we're comparing $\log_2 3$ against $\frac{p}{q}$ for some fraction $\frac{p}{q}$. Is $\log_2 3 < \frac{p}{q}$? Well, if it is, then $3 < 2^{p/q}$, so $3^q < 2^p$. You've got particular values of p and q in mind; go for it!

  You might like to reflect on why it's OK to manipulate the inequalities like this.

MAT 2012 Q1C

- Simplify each number as much as you can before doing any comparisons.

Extension

[Just for fun, not part of the MAT question]

- Given a positive number α, which is the smallest of these values? Identify the different cases according to α.

$$\text{(a)}\quad \alpha, \qquad \text{(b)}\quad \sqrt{2\alpha}, \qquad \text{(c)}\quad \alpha^{-3} \qquad \text{(d)}\quad \frac{2}{\alpha}.$$

- Which is larger, $(8!)^9$ or $(9!)^8$?