JMAT 7304

Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Linear Algebra & Finite Element Methods

TRINITY TERM 2014

Friday 25th April 2014, 9.30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Part A — Numerical Linear Algebra

Question 1

Throughout this question we consider A to be an $m \times m$ real matrix. We denote the error and residual of the n^{th} iterate of an iterative method for solving $Ax = b$ by $e_n = x - x_n$ and $r_n = b - Ax_n$ respectively. The variables α_n, and β_n in parts (a), (b), and (c) of this question are distinct and unrelated.

(a) Consider an iterative method of the form

$$x_{n+1} = x_n + \alpha_n A^*(b - Ax_n)$$

to compute approximate solutions to $Ax = b$ where A^* is the transpose of A. Determine a formula for α_n so that the error e_{n+1} is minimized in the ℓ^2 norm. Determine a bound on the error of the form

$$\|e_{n+1}\|_2 \leq \delta \|e_n\|_2,$$

where δ is a function of the condition number of A. Contrast this method and its convergence rate with that of the steepest descent method.

[9 marks]

(b) Consider an iterative method of the form

$$\begin{aligned} x_{n+1} &= x_n + \alpha_n p_n \\ p_{n+1} &= r_{n+1} + \beta_n p_n \end{aligned}$$

with the initial p_0 defined as $p_0 = r_0$. Determine a formula for α_n so that the residual r_{n+1} is minimized in the ℓ^2 norm and determine a formula for β_n so that Ap_{n+1} and Ap_n are orthogonal in the ℓ^2 inner product. Explain why it is beneficial to define β_n in this way.

[9 marks]

(c) The conjugate gradient method is defined by

$$\begin{aligned} x_{n+1} &= x_n + \alpha_n p_n \\ p_{n+1} &= r_{n+1} + \beta_n p_n \end{aligned}$$

with $\alpha_n = \frac{r_n^* p_n}{p_n^* A p_n}$ and $\beta_n = \frac{-r_{n+1}^* A p_n}{p_n^* A p_n}$. Show that the residuals and search directions satisfy $r_{n+1}^* r_j = 0$ and $p_{n+1}^* A p_j = 0$ for all $j \leq n$. Use these properties to determine the error at the m^{th} iteration of the conjugate gradient method applied to an $m \times m$ system of equations.

[7 marks]

Question 2

(a) State the power method for computing the eigenvalue of an invertible matrix A that is the *smallest* in magnitude. Prove that if the matrix is Hermitian positive definite, then the power method converges to the eigenvalue of *smallest* magnitude. Also establish the rate of convergence.

[8 marks]

(b) Consider a matrix A whose eigenvalue of greatest magnitude, denoted by λ, is repeated $r > 1$ times, and suppose that A *does not* have r linearly independent eigenvectors associated with the eigenvalue λ. Does the power method to compute the eigenvalue of largest magnitude applied to this matrix converge to the largest eigenvalue? The general case need not be proven; illustrate your answer by constructing a simple 3×3 example.

[5 marks]

(c) Simultaneous Iteration is an iterative methods for computing the eigenvectors of a matrix A, and is given by: Set $\hat{Q}^{(0)} = I$, the identity matrix, and iterate

for $k = 1, 2, \ldots,$
$Z^{(k)} = A\hat{Q}^{(k-1)}$
$\hat{Q}^{(k)}\hat{R}^{(k)} = Z^{(k)}$, the QR decomposition of $Z^{(k)}$.

Show that $\hat{Q}^{(k)}\hat{R}^{(k)}\hat{R}^{(k-1)}\hat{R}^{(k-2)} \cdots \hat{R}^{(1)}$ is equal to the k^{th} power of A, that is A^k. Explain the role of the product $\hat{R}^{(k)}\hat{R}^{(k-1)}\hat{R}^{(k-2)} \cdots \hat{R}^{(1)}$ and the role of $\hat{Q}^{(k)}$ in this algorithm.

[7 marks]

(d) Let A be diagonalizable with eigenvalue of largest magnitude given by μ_1 and let ν_1 be the associated eigenvector, so that $A\nu_1 = \mu_1\nu_1$. Use the results from Part (c) of this question to deduce that if the first column of A is not orthogonal to ν_1 then the first column of $\hat{Q}^{(k)}$ converges to a multiple of ν_1 and state the rate of convergence. How would this result change if the first column of A were orthogonal to ν_1?

[5 marks]

Section B — Finite Element Methods

Question 3

(a) Suppose that $f \in \mathrm{L}^2(0,1)$. State the weak formulation of the boundary-value problem

$$\begin{aligned} -u'' + (1-x)u' &= f(x), \qquad x \in (0,1), \\ u'(0) &= u(0), \quad u'(1) = 0. \end{aligned}$$

Show that the bilinear form associated with the weak formulation of this problem is coercive on $\mathrm{H}^1(0,1)$.

By using the Lax–Milgram Theorem, show that the boundary-value problem has a unique weak solution u in $\mathrm{H}^1(0,1)$.

[The following inequality may be used without proof:

$$w^2(0) \le \|w\|^2_{\mathrm{L}^2(0,1)} + 2\|w\|_{\mathrm{L}^2(0,1)}\|w'\|_{\mathrm{L}^2(0,1)}, \qquad w \in \mathrm{H}^1(0,1).]$$

[9 marks]

(b) Consider the continuous piecewise linear basis functions φ_i, $i = 0, 1, \ldots, N$, defined by $\varphi_i(x) = (1 - |x - x_i|/h)_+$ on the uniform mesh of size $h = 1/N$, $N \ge 2$, with mesh-points $x_i = ih$, $i = 0, 1, \ldots, N$.

Using the basis functions φ_i, $i = 0, 1, \ldots, N$, define the finite element approximation of the boundary-value problem and show that it has a unique solution u_h.

Expand u_h in terms of the basis functions φ_i, $i = 0, 1, \ldots, N$, by writing

$$u_h(x) = \sum_{i=0}^{N} U_i \varphi_i(x)$$

where $\mathbf{U} = (U_0, U_1, \ldots, U_N)^{\mathrm{T}} \in \mathbb{R}^{N+1}$, to obtain a system of linear algebraic equations for the vector of unknowns $\mathbf{U}$.

Show that the matrix $\mathcal{A}$ of this linear system is nonsingular.

[9 marks]

(c) Show that there exists a positive constant C, independent of h, such that, for any continuous piecewise linear function v_h on a subdivision of $[0,1]$ defined by the mesh-points x_i, $i = 0, 1, \ldots, N$,

$$\|u - u_h\|_{\mathrm{H}^1(0,1)} \le C\|u - v_h\|_{\mathrm{H}^1(0,1)}.$$

Deduce that $\|u - u_h\|_{\mathrm{H}^1(0,1)} = \mathcal{O}(h)$ as $h \to 0$.

[7 marks]

[*You may assume that the unique weak solution of the boundary-value problem belongs to* $\mathrm{H}^2(0,1)$. *Any bound on the error between* u *and its finite element interpolant* $\mathcal{I}_h u$ *may be used without proof, but must be stated carefully.*]

Question 4

Suppose that $\Omega = (0,1)^2$ and $f \in \mathrm{L}^2(\Omega)$. Consider the quadratic energy functional $J : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}\int_\Omega \left[\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y}\right)^2 + v^2 - 2fv\right] \mathrm{d}x\,\mathrm{d}y.$$

(a) Show that if $u \in \mathrm{H}_0^1(\Omega)$ is such that

$$J(u) \le J(v) \qquad \text{for all } v \in \mathrm{H}_0^1(\Omega),$$

then there exists a bilinear functional $a : \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ and a linear functional $\ell : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ such that

$$a(u,v) = \ell(v) \qquad \text{for all } v \in \mathrm{H}_0^1(\Omega).$$

[9 marks]

Show further that:

(i) $a(w,v) = a(v,w)$ for all $w, v \in \mathrm{H}_0^1(\Omega)$;

(ii) $a(v,v) \ge \frac{1}{2}\|\nabla v\|^2_{\mathrm{L}^2(\Omega)} + \|v\|^2_{\mathrm{L}^2(\Omega)}$ for all $v \in \mathrm{H}_0^1(\Omega)$;

(iii) $|a(w,v)| \le \frac{3}{2}\|\nabla w\|_{\mathrm{L}^2(\Omega)}\|\nabla v\|_{\mathrm{L}^2(\Omega)} + \|w\|_{\mathrm{L}^2(\Omega)}\|v\|_{\mathrm{L}^2(\Omega)}$ for all $w, v \in \mathrm{H}_0^1(\Omega)$.

[6 marks]

(b) Consider a triangulation of $\overline{\Omega}$ which has been obtained from a square mesh of spacing $h = 1/N$, $N \ge 2$, in both co-ordinate directions by subdividing each mesh-square into two triangles with the diagonal on negative slope. Denote by V_h the finite-dimensional subspace of $\mathrm{H}_0^1(\Omega)$ consisting of piecewise linear functions defined on this triangulation.

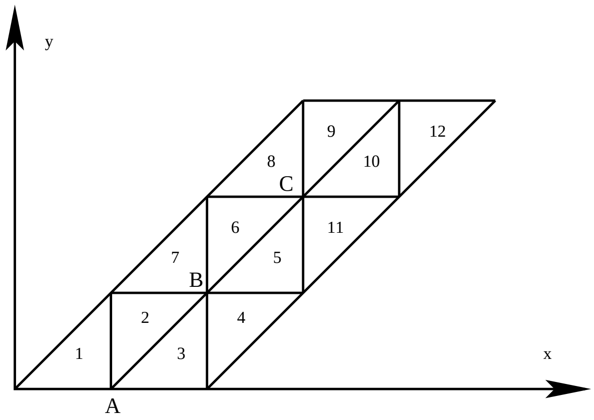
Show that there exists a unique element u_h in V_h such that $J(u_h) \le J(v_h)$ for all $v_h \in V_h$.

[10 marks]

Question 5

Suppose that $\Omega = \{(x, y) \in \mathbb{R}^2 \, : \, y < x < y + 2, \ 0 < y < 3\}$.

(a) Consider the triangulation $\mathcal{T}$ of the computational domain $\overline{\Omega}$ defined by the lines $y = x$, $y = x - 1$, $y = x - 2$, $y = 0$, $y = 1$, $y = 2$, $y = 3$, $x = 1$, $x = 2$, $x = 3$, $x = 4$, as shown in the figure, and the nodes $\mathsf{A} = (1, 0)$, $\mathsf{B} = (2, 1)$, $\mathsf{C} = (3, 2)$, as indicated.



Define the continuous piecewise linear finite element basis functions φ_{A}, φ_{B} and φ_{C} associated with the nodes A, B and C, carefully specifying their values in each of the 12 triangles of the triangulation $\mathcal{T}$, indicated by the numbers 1,2,. . . ,12 in the figure.

[10 marks]

(b) Let $\Gamma_{\mathrm{N}} = \{(x, 0) \in \mathbb{R}^2 \, : \, 0 \le x \le 2\}$, and $\Gamma_{\mathrm{D}} = \partial\Omega \setminus \Gamma_{\mathrm{N}}$. State the weak formulation of the elliptic boundary value problem

$$
\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} &= 1 \qquad \text{in } \Omega, \\
\frac{\partial u}{\partial y} &= 0 \qquad \text{on } \Gamma_{\mathrm{N}}, \\
u &= 0 \qquad \text{on } \Gamma_{\mathrm{D}}.
\end{aligned}
$$

State the continuous piecewise linear finite element approximation of the boundary value problem, defined on the triangulation $\mathcal{T}$.

[5 marks]

(c) Recast the finite element method as a system of linear algebraic equations, and compute the values of the finite element approximation to the function u at the nodes A, B and C.

[10 marks]

Question 6

Let $u = u(x,t)$ denote the solution to the initial-boundary-value problem

$$\begin{aligned} (1+x)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad 0 < t \le T, \\ u(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0, \qquad 0 \le t \le T, \\ u(x,0) = u_0(x), \qquad 0 < x < 1, \end{aligned}$$

where $T > 0$, $u_0 \in \mathrm{L}^2(0,1)$.

(a) Construct a finite element method for the numerical solution of this problem, based on the implicit Euler scheme with time step $\Delta t = T/M$, $M \ge 2$, and a piecewise linear approximation in x on a uniform subdivision of spacing $h = 1/N$, $N \ge 2$, of the interval $[0,1]$, denoting by u_h^m the finite element approximation to $u(\cdot, t^m)$ where $t^m = m\Delta t$, $0 \le m \le M$.

[9 marks]

(b) Consider the inner product

$$(w,v)_\star = \int_0^1 (1+x)w(x)v(x)\,\mathrm{d}x$$

and the associated norm $\|\cdot\|_\star$ defined by $\|w\|_\star^2 = (w,w)_\star$.

Show that, for $0 \le m \le M-1$,

$$\frac{1}{2\Delta t}\left(\|u_h^{m+1}\|_\star^2 - \|u_h^m\|_\star^2\right) + \frac{1}{2\Delta t}\left\|u_h^{m+1} - u_h^m\right\|_\star^2 + \left|u^{m+1}\right|_{\mathrm{H}^1(0,1)}^2 = 0,$$

where $|\cdot|_{\mathrm{H}^1(0,1)}$ is the seminorm of the Sobolev space $\mathrm{H}^1(0,1)$.

Hence deduce that the method is unconditionally stable in the $\|\cdot\|_\star$ norm in the sense that, for any Δt, independent of the choice of h,

$$\|u_h^m\|_\star \le \|u_h^0\|_\star, \qquad 1 \le m \le M.$$

[9 marks]

(c) Show that for each m, $0 \le m \le M-1$, u_h^{m+1} can be obtained from u_h^m by solving a system of linear algebraic equations with a symmetric matrix $\mathcal{A}$ whose entries you should define in terms of the standard piecewise linear basis functions φ_i, $i = 1, \dots, N$. Assuming that $N = 2$ and $\Delta t = 1$, compute the *off-diagonal* entries of $\mathcal{A}$.

[7 marks]