Delamination and fracture of thin films

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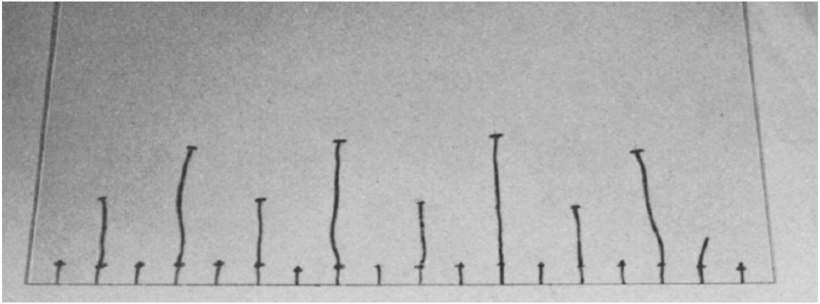
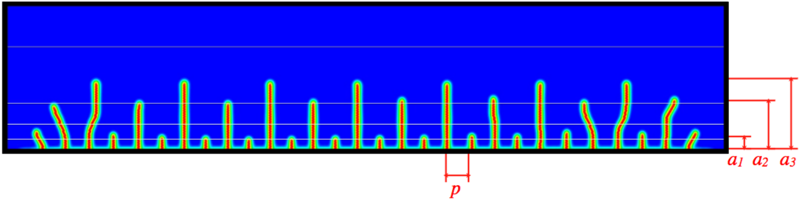
Brittle fracture in thin films

Geyer & Nemat-Nasser (IJSS 1982)



Brittle fracture

Bourdin & Maurini (in preparation)



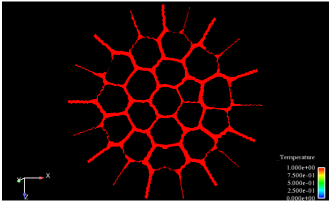
$$\mathcal{E}(\mathbf{u}, K) = \int_{\Omega \setminus K} \frac{1}{2}\mathbf{A}(\mathbf{e}(\mathbf{u}) - \theta\mathbf{I}) \cdot (\mathbf{e}(\mathbf{u}) - \theta\mathbf{I}) + G_c \mathcal{H}^{n-1}(K)$$

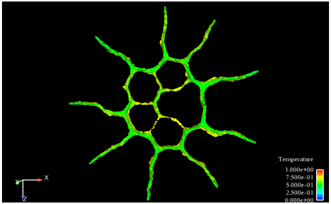
approximated by

$$\mathcal{E}_\ell(\mathbf{u}, \alpha) = \int_{\Omega} \frac{1}{2}(1-\alpha)^2\mathbf{A}(\mathbf{e}(\mathbf{u}) - \theta\mathbf{I}) \cdot (\mathbf{e}(\mathbf{u}) - \theta\mathbf{I}) + \int_{\Omega} \left(\frac{9\alpha}{64\ell} + \ell|\nabla\alpha|^2 \right)$$

Thermal shock on a cylinder

Bourdin & Maurini (in preparation)





Thermal shock on a cylinder

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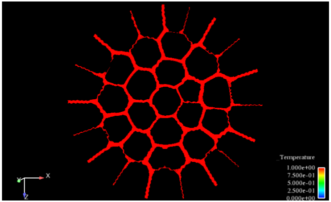
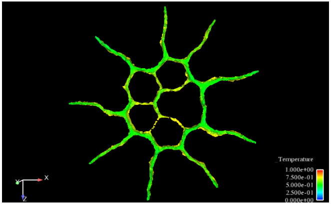
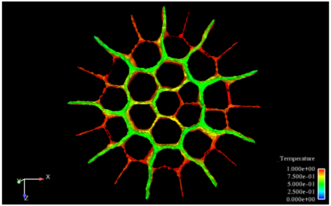




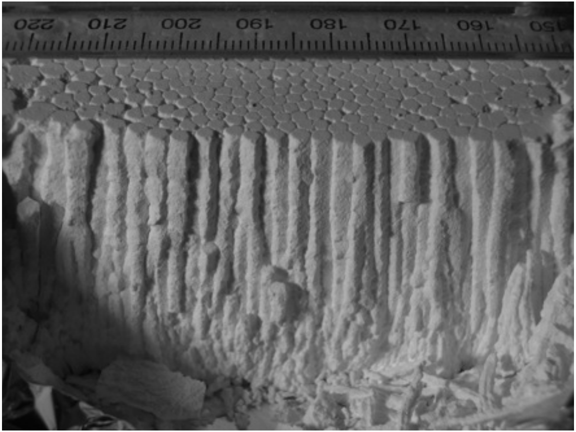


Thermal cracks

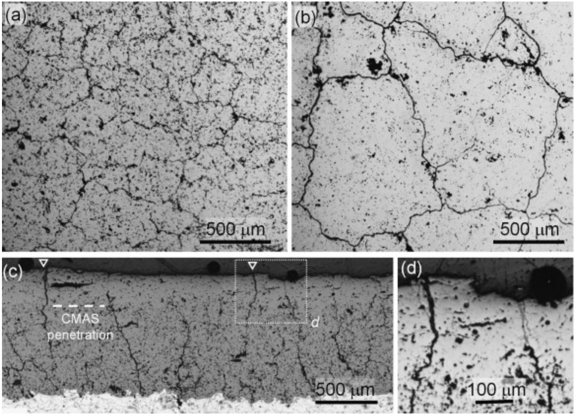
Goehring, Mahadevan & Morris (PNAS 2009)

Thermal cracks

Goehring, Mahadevan & Morris (PNAS 2009)

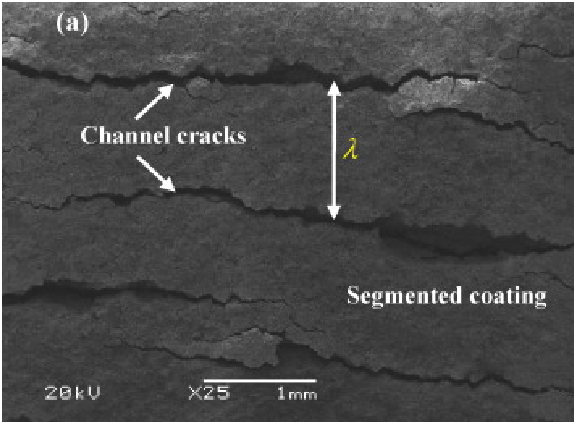


Krämer et al. (MSE 2008)



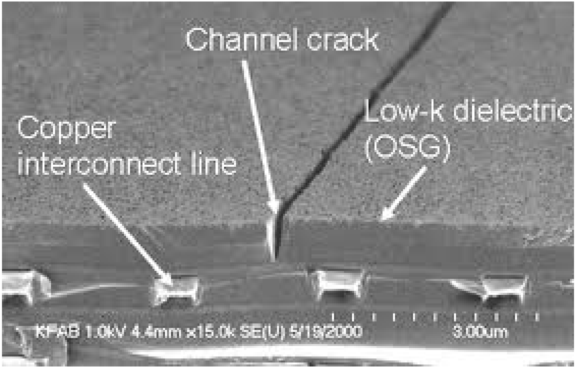
Cracks in thin films

Wu et al. (ASS 2011)



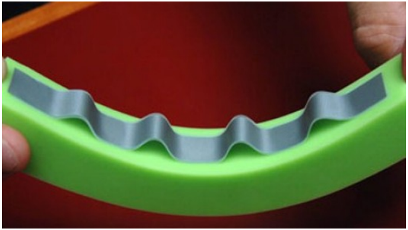
Cracks in thin films

Tsui, McKerrow & Vlassak (JMR 2005)



Delamination

Reis (PNAS 2009)



Airwolf Aerospace

Delamination & Fracture

León Baldelli et al. (in preparation)

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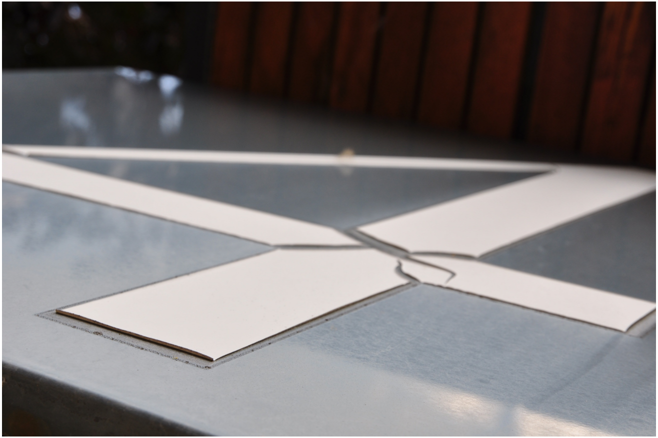
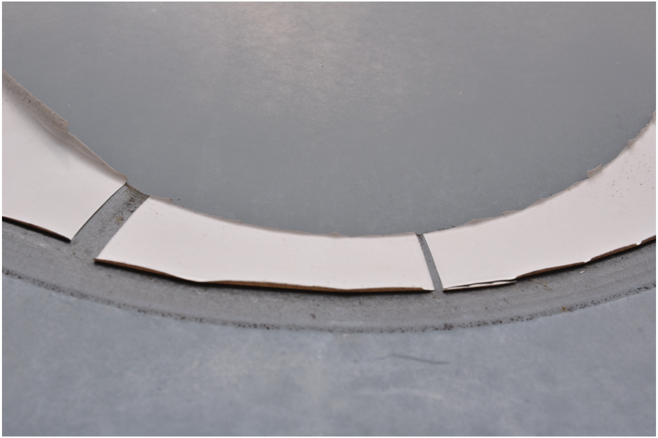


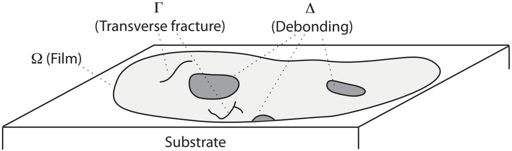
Delamination & Fracture

León Baldelli et al. (in preparation)



Delamination & Fracture

León Baldelli, Bourdin, Marigo & Maurini (CMT 2012)



$$\min_{\substack{\mathbf{u}' \in SBD(\omega;\mathbb{R}^2)\\ \Delta\subset\omega}} \frac{1}{2}\int_\omega A_\infty(\mathbf{e}(\mathbf{u}') - \theta\mathbf{I}_{2\times 2})\cdot(\mathbf{e}(\mathbf{u}') - \theta\mathbf{I}_{2\times 2})$$

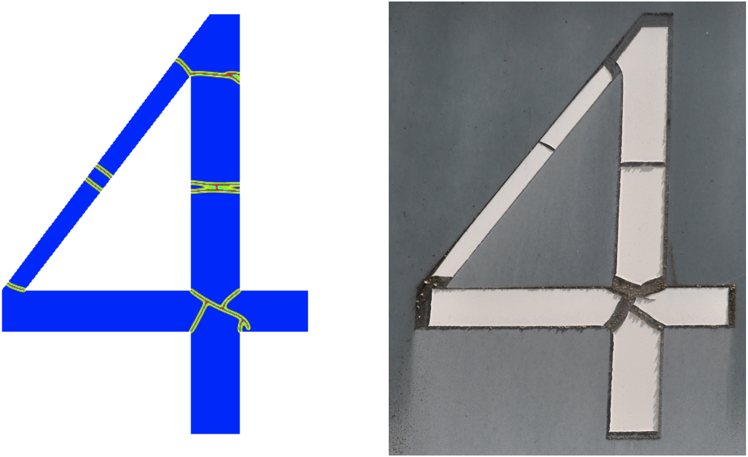
$$+ \quad \mathcal{H}^1(J_{\mathbf{u}'}) \quad + \quad \frac{\mu'}{2}\int_{\omega\setminus\Delta}\mathbf{u}'^2 \quad + \kappa\mathcal{H}^2(\Delta)$$

with

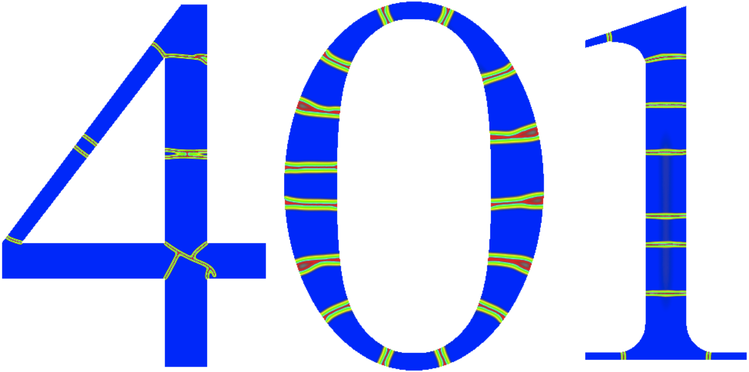
$$\mathbf{A}_\infty\mathbf{e} = \frac{2\lambda\mu}{\lambda+2\mu} e_{\alpha\alpha}e_{\beta\beta} + 2\mu e_{\alpha\beta}e_{\alpha\beta}$$

Delamination & Fracture

León Baldelli, Bourdin, Maurini



Delamination & Fracture



Linear plate theory (e.g. Ciarlet vol. II)

- $\Omega_\varepsilon = \omega \times (0,\varepsilon) \subset \mathbb{R}^3$
- $\mathbf{v}_\varepsilon = (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon) \in H^1(\Omega_\varepsilon \times (0,\varepsilon))$
- $\displaystyle \frac{1}{2}\int_{\Omega_\varepsilon} 2\mu \left| \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \right|^2 + \lambda(e_{\alpha\alpha} + e_{33})^2$
- $v_3^\varepsilon(x_1, x_2, \varepsilon x_3) = \varepsilon u_3^\varepsilon(x_1, x_2, x_3)$
- $v_\alpha^\varepsilon(x_1, x_2, \varepsilon x_3) = \varepsilon^2 u_\alpha^\varepsilon(x_1, x_2, x_3)$

$$\Gamma - \text{lím} = \frac{1}{2}\int_\Omega \frac{2\lambda\mu}{\lambda + 2\mu} e_{\alpha\alpha}(\mathbf{u}) e_{\beta\beta}(\mathbf{u}) + 2\mu e_{\alpha\beta}(\mathbf{u}) e_{\alpha\beta}(\mathbf{u}),$$

$$e_{\alpha 3}(\mathbf{u}) = e_{33}(\mathbf{u}) = 0.$$

Dimension reduction

- Le Dret & Raoult, JMPA '95: *nonlinear membrane model*
- Fonseca & Francfort, JRAM '98: *3D-2D in optimal design*
- Bhattacharya & James, JMPS '99: *martensitic thin films*
- Braides, Fonseca & Francfort, IUMJ '00: *inhomogeneous thin films*
- Mora & Scardia, JDE '12: *covergence of equilibria of physical plates*
- ...

Brittle thin films

Braides & Fonseca (AMP 2001),

Bouchitté, Fonseca, Leoni & Mascarenhas (ARMA 2002):

- $\Omega_\varepsilon = \omega \times (0, \varepsilon)$, $\mathbf{u} \in GSBV_p(\Omega; \mathbb{R}^3)$,
- $J_\varepsilon = \displaystyle\int_\Omega W\left(\nabla_\alpha \mathbf{u} \middle| \frac{1}{\varepsilon}\nabla_3 \mathbf{u}\right) + \int_{J_\mathbf{u}} \vartheta\left(\mathbf{u}^+ - \mathbf{u}^-, \boldsymbol{\nu}_\alpha(\mathbf{u}), \frac{1}{\varepsilon}\nu_3(\mathbf{u})\right) \mathrm{d}\mathcal{H}^2$
- ϑ symmetric, positively 1-homogeneous, Lipschitz, linear at infinity
- $|F|^p \leq W(F) \leq C(1 + |F|^p)$, continuous

Γ-converges to

$$\int_\omega Q\bar{W}(\nabla\alpha\mathbf{u}) + \int_{J_\mathbf{u}\cap\omega} R\bar{\vartheta}(\mathbf{u}^+ - \mathbf{u}^-, \boldsymbol{\nu}_\alpha(\mathbf{u}))\,\mathrm{d}\mathcal{H}^1$$

$\mathbf{u} \in SBV_p(\Omega; \mathbb{R}^3) : \nabla_3\mathbf{u} = 0, \nu_3(\mathbf{u}) = 0.$

Brittle thin films

- **Giacomini CVPDE '05:**
  Ambrosio-Tortorelli for the quasistatic evolution

- **Babadjian CVPDE '06:**
  Convergence of the quasistatic evolutions

Variational delamination

Bhattacharya, Fonseca & Francfort (ARMA 2002):

- $\Omega_\varepsilon = \omega \times (-\varepsilon s, \varepsilon h)$, $\mathbf{u} \in W^{1,p}(\Omega^+; \mathbb{R}^3) \cap W^{1,p}(\Omega^-; \mathbb{R}^3)$,
- $\displaystyle\int_{\Omega^+} W^+\left(\nabla_\alpha \mathbf{u} \middle| \frac{1}{\varepsilon}\nabla_3 \mathbf{u}\right) + \int_{\Omega^-} W^-\left(\nabla_\alpha \mathbf{u} \middle| \frac{1}{\varepsilon}\nabla_3 \mathbf{u}\right) + \varepsilon^{\alpha-1}\int_\omega |[\mathbf{u}]|^\gamma$
- $\frac{1}{C}|F|^p - C \leq W(F) \leq C(1+|F|^p)$, continuous

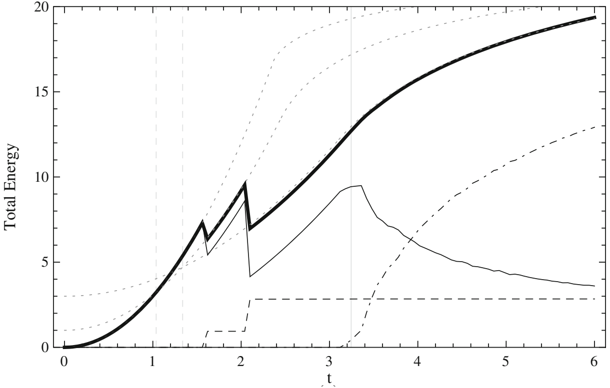
Γ-converges to $\int_\omega hQ\overline{W}^+ + sQ\overline{W}^-$ (small debonding with small energy, independent oscillations in each layer; in the limit $\mathbf{u}$ is continuous and independent of x_3).

Cohesive interfacial energy $\int_\omega |[u]|^y$

- **Ansini AA '04:** *nonlinear Neumann sieve*
- **Ansini, Babadjian & Zeppieri M3AS '07:** *multiscale Neumann sieve*
  - Ansini-Braides JMPA '02, AAP '01; Attouch-Picard RSMUPT '87; Conca JMPA 1985, 1987; Damlamian RDMUPT '85; Del Vecchio AMPA '87; Murat '85; Sanchez-Palencia '80, '81, '85
- **Roubíček, Scardia, Zanini CMT '09:** $\displaystyle\int_{\Gamma_c} kz[\mathbf{u}]^2_{\Gamma_c}\,\mathrm{d}\mathcal{H}^{n-1}$
- **Freddi, Paroni, Roubíček & Zanini ZAMM '11:** transverse fracture as a form of delamination

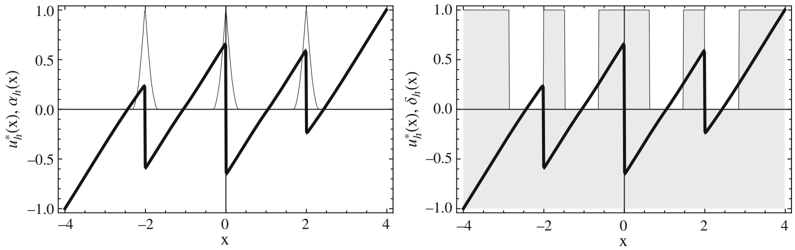
León Baldelli, Bourdin, Marigo & Maurini CMT '12:

$$\int_{\omega\setminus\Gamma} \frac{1}{2}(u'(x)-t)^2\,\mathrm{d}x + \int_{\Omega\setminus\Gamma} \frac{1}{2}u(x)^2\,\mathrm{d}x + \#(\Gamma) + \gamma\mathcal{H}^1(\Delta)$$

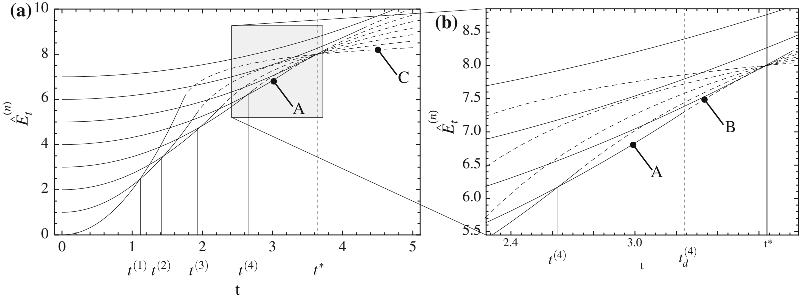


1D delamination and debonding

León Baldelli, Bourdin, Marigo & Maurini CMT '12:



León Baldelli, Bourdin, Marigo & Maurini CMT '12



Multi-layer asymptotic analysis

- $\Omega = \omega \times (0,1)$, $\Omega' = \omega \times (-1,0)$; $\omega \subset \mathbb{R}^2$
- **Vanishing Young's modulus** in the bonding layer

$$(\lambda_\varepsilon, \mu_\varepsilon) = \begin{cases} (\lambda, \mu) & \text{in } \Omega \\ \varepsilon^2 (\lambda', \mu') & \text{in } \Omega' \end{cases}$$

- Rescaled energies

$$J_\varepsilon(\mathbf{u}, \Omega) = \frac{1}{2} \int_\Omega 2\mu \left| \begin{pmatrix} e_{11} & e_{12} & \varepsilon^{-1} e_{13} \\ e_{21} & e_{22} & \varepsilon^{-1} e_{13} \\ \varepsilon^{-1} e_{31} & \varepsilon^{-1} e_{32} & \varepsilon^{-2} e_{33} \end{pmatrix} \right|^2 + \lambda (e_{\alpha\alpha} + \varepsilon^{-1} e_{33})^2$$

$$J_\varepsilon(\mathbf{u}, \Omega') = \frac{1}{2} \int_{\Omega'} 2\mu \left| \begin{pmatrix} \varepsilon e_{11} & \varepsilon e_{12} & e_{13} \\ \varepsilon e_{21} & \varepsilon e_{22} & e_{13} \\ e_{31} & e_{32} & \varepsilon^{-1} e_{33} \end{pmatrix} \right|^2 + \lambda (e_{\alpha\alpha} + \varepsilon^{-1} e_{33})^2$$

- $\mathbf{u} \in H^1(\Omega \cup \Omega')$, $\mathbf{u}(\cdot, -1) = 0$

Multi-layer asymptotic analysis

- Energies:

$$J_\varepsilon(\mathbf{u},\Omega) = \frac{1}{2}\int_\Omega 2\mu \left| \begin{pmatrix} e_{11} & e_{12} & \varepsilon^{-1}e_{13} \\ e_{21} & e_{22} & \varepsilon^{-1}e_{13} \\ \varepsilon^{-1}e_{31} & \varepsilon^{-1}e_{32} & \varepsilon^{-2}e_{33} \end{pmatrix} \right|^2 + \lambda(e_{\alpha\alpha} + \varepsilon^{-1}e_{33})^2$$

$$J_\varepsilon(\mathbf{u},\Omega') = \frac{1}{2}\int_{\Omega'} 2\mu \left| \begin{pmatrix} \varepsilon e_{11} & \varepsilon e_{12} & e_{13} \\ \varepsilon e_{21} & \varepsilon e_{22} & e_{13} \\ e_{31} & e_{32} & \varepsilon^{-1}e_{33} \end{pmatrix} \right|^2 + \lambda(e_{\alpha\alpha} + \varepsilon^{-1}e_{33})^2$$

- $\mathbf{u} \in H^1(\Omega \cup \Omega')$, $\mathbf{u}(\cdot,-1) = 0$, then

Multi-layer asymptotic analysis

- Energies:

$$J_\varepsilon(\mathbf{u},\Omega)=\frac{1}{2}\int_\Omega 2\mu\left|\begin{pmatrix} e_{11} & e_{12} & \varepsilon^{-1}e_{13}\\ e_{21} & e_{22} & \varepsilon^{-1}e_{13}\\ \varepsilon^{-1}e_{31} & \varepsilon^{-1}e_{32} & \varepsilon^{-2}e_{33}\end{pmatrix}\right|^2+\lambda(e_{\alpha\alpha}+\varepsilon^{-1}e_{33})^2$$

$$J_\varepsilon(\mathbf{u},\Omega')=\frac{1}{2}\int_{\Omega'} 2\mu\left|\begin{pmatrix} \varepsilon e_{11} & \varepsilon e_{12} & e_{13}\\ \varepsilon e_{21} & \varepsilon e_{22} & e_{13}\\ e_{31} & e_{32} & \varepsilon^{-1}e_{33}\end{pmatrix}\right|^2+\lambda(e_{\alpha\alpha}+\varepsilon^{-1}e_{33})^2$$

- $\mathbf{u}\in H^1(\Omega\cup\Omega')$, $\mathbf{u}(\cdot,-1)=0$, then

- **Theorem**

$J_\varepsilon(\mathbf{u},\Omega)+J_\varepsilon(\mathbf{u},\Omega')$ Γ-*converges to*

$$\frac{1}{2}\int_\omega\left[\frac{2\lambda\mu}{\lambda+2\mu}e_{\alpha\alpha}e_{\beta\beta}+2\mu e_{\alpha\beta}e_{\alpha\beta}\right]+\frac{\mu'}{2}\int_\omega u_\alpha u_\alpha$$

Delamination and fracture

For $\mathbf{u} \in SBV(\Omega \cup \Omega' \cup \Omega''; \mathbb{R}^3)$, $\mathbf{u} = \mathbf{w}$ a.e. in Ω'', define

$$\mathcal{F}_\varepsilon(\mathbf{u}) = \int_\Omega \left(|\nabla' \mathbf{u} - \mathbf{A}_0|^2 + \varepsilon^{-2} |\partial_3 \mathbf{u}|^2 \right) \mathrm{d}x + \int_{\Omega'} \left(\varepsilon^2 |\nabla' \mathbf{u}|^2 + |\partial_3 \mathbf{u}|^2 \right) \mathrm{d}x$$
$$+ \int_{\Omega \cap J_\mathbf{u}} \left| \left((\boldsymbol{\nu}_\mathbf{u})', \frac{1}{\varepsilon} (\boldsymbol{\nu}_\mathbf{u})_3 \right) \right| \mathrm{d}\mathcal{H}^2 + \int_{\Omega' \cap J_\mathbf{u}} |(\varepsilon (\boldsymbol{\nu}_\mathbf{u})', (\boldsymbol{\nu}_\mathbf{u})_3)| \, \mathrm{d}\mathcal{H}^2.$$

Theorem

$\mathcal{F}_\varepsilon$ converges to

$$\int_\omega |\nabla' \mathbf{u} - \mathbf{A}_0|^2 \, \mathrm{d}\mathbf{x}' + \int_{\{|\mathbf{u}-\mathbf{w}| \leq 1\}} |\mathbf{u} - \mathbf{w}|^2 \, \mathrm{d}\mathbf{x}'$$
$$+ \mathcal{H}^1(J_\mathbf{u}) + \mathcal{H}^2(\{|\mathbf{u} - \mathbf{w}| > 1\}),$$

for $\mathbf{u} \in SBV(\omega, \mathbb{R}^3)$ *and* $\max_\alpha \|u_\alpha\|_{L^\infty} \leq M$.