Linear and non linear Calderón-Zygmund theories

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1 - The classics

A model case

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otherwise failing when

$$q = 1, \infty$$

The classical approach

- **We use the fundamental solution**

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- Differentiating **twice** yields

$$D^2u(x) = \int K(x,y)\mu(y)\,dy$$

where $K(x,y)$ is CZ singular kernel

Singular kernels - cancelations

- $T\colon \mu \mapsto \int K(x,y)\mu(y)\,dy$

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$$\|\hat{K}\|_{L^\infty} \leq B\,,$$

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Singular kernels - cancelations

- $T \colon \mu \mapsto \int K(x,y)\mu(y)\,dy$
- **Well-posedness** L^2

$$\|\hat{K}\|_{L^\infty} \leq B\,,$$

where $\hat{K}$ denotes the Fourier transform $K(\cdot)$

- **Hörmander's cancelation condition**

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)|\,dx \leq B \qquad \text{for every} \quad y \in \mathbb{R}^n$$

- **then**

$$T \colon L^q \to L^q \qquad\qquad 1 < q < \infty$$

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- for which it holds

$$I_\beta \colon L^q \to L^{\frac{nq}{n-\beta q}} \qquad \beta q < n$$

When size counts

- Singular integrals need finer analysis as delicate cancellation properties come into the play; for fractional integrals analyzing the size of the kernel suffices

Another linear case

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- A subsequent approach via linear interpolation has been given by Campanato and Stampacchia

2 - The dual case

Nonlinear, degenerate cases

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- the solution $u\colon \Omega \to \mathbb{R}^N$ is in a Sobolev space
- **and we are considering the standard ellipticity assumptions**

$$\begin{cases} |a(z)| + |\partial a(z)||z| \leq L|z|^{p-1} \\ \nu|z|^{p-2}|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda\rangle \end{cases}$$

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- when $H = 0$ the previous one is the Euler-Lagrange equation of the functional

$$w \mapsto \int_\Omega |Dw|^p \, dx$$

The setting

- The classical distributional formulation reads as

$$\int_\Omega \langle a(Du), D\varphi\rangle \, dx = \langle H, \varphi\rangle \qquad \forall\ \varphi \in C^\infty$$

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- Basic issues
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- Where to find u?
- These are related questions; monotone operators theory tells we can find a solutions in $W^{1,p}$ when H belongs to the dual $W^{-1,p'}$
- At this stage we are dealing with **energy** solutions $\varphi \approx u$

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- Weak formulations as

$$\int_\Omega \langle a(Du), D\varphi \rangle \, dx = \langle H, \varphi \rangle \qquad \forall \ \varphi \in C^\infty$$

still make sense when u is not that integrable and H does not necessarily belong to the dual; it suffices that $Du \in L^{p-1}$

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- and in fact it is possible to find solutions which do not belong to $W^{1,p}$, these are called **very weak solutions**
- In this first part I will deal with the case H belongs to the dual

$$H = -\mathrm{div}\,(|F|^{p-2}F)\,, \qquad F \in L^q\,, \qquad q \geq p$$

The fundamentals

Theorem (Iwaniec, Studia Math. 83)

If u solves

$$\operatorname{div}\,(|Du|^{p-2}Du) = \operatorname{div}\,(|F|^{p-2}F) \qquad \textit{in } \mathbb{R}^n$$

then

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Theorem (DiBenedetto & Manfredi, Amer. J. Math. 93)

The previous result still holds in the case of systems; moreover

$$F \in BMO \Longrightarrow Du \in BMO$$

Subsequent contributions

- **Caffarelli & Peral (1998)** give an important, yet maximal function based, approach to the L^p estimates in the setting of homogenization problems

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Subsequent contributions

- **Caffarelli & Peral (1998)** give an important, yet maximal function based, approach to the L^p estimates in the setting of homogenization problems
- **Krylov (2005), in a series of papers**, employs maximal operators to obtain parabolic estimates
- **Subsequent contributions have been given by several authors:** Byun, Diening, Dong, Gutierrez, Iwaniec, Kim, Kinnunen, Kristensen, L. Wang, Peral, Sbordone, Scheven, Yao, Zhou

- **Iwaniec's result extends**

$$\text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

More general operators

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$$\text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

under the standard ellipticity conditions

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- and to systems of the type

$$\text{div } (g(|Du|)Du) = \text{div } (|F|^{p-2}F)$$

More general operators

- **Iwaniec's does not extend to general systems of the type**

$$\operatorname{div}\, a(Du) = \operatorname{div}\, (|F|^{p-2}F)$$

with $u\colon \Omega \to \mathbb{R}^N$ and $N > 1$

- Indeed, **a fundamental result of Šverák & Yan** claims the existence of unbounded solutions for homogenous systems of the type

$$\operatorname{div}\, a(Du) = 0$$

More general operators

Anyway something survives

Theorem (Kristensen & Min., ARMA 06)

For solutions

$$\mathrm{div}\ a(Du) = \mathrm{div}\ (|F|^{p-2}F)$$

we have

$$F \in L^q \Longrightarrow Du \in L^q \qquad p \leq q < \infty$$

whenever

$$p < q < \frac{np}{n-2} + \delta$$

- For more on the nature of δ see a basic work of **Kristensen & Melcher** (Math. Z. 08)

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- This result turns out to be crucial in order to get certain precise estimates for singular sets of solutions to vectorial problems

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Parabolic equations

Theorem (Acerbi & Min., Duke Math. J. 07)

If u solves

$$u_t - \operatorname{div}\,(|Du|^{p-2}Du) = \operatorname{div}\,(|F|^{p-2}F) \qquad \textit{in } \Omega \times (0,T)$$

and

$$p > \frac{2n}{n+2}$$

then

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The lower bound

$$p > \frac{2n}{n+2}$$

is optimal

Parabolic equations

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- Crucial use of **DiBenedetto's regularity estimates**
- **The result extends to systems**
- **The result extends to general equations of the type**

$$u_t - \text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

An elementary approach

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- New, interpolation free proof of classical Calderón-Zygmund theory

Elementary approach

- For the result

$$\triangle u = \operatorname{div} F \qquad \Rightarrow \qquad \|Du\|_{L^q} \lesssim \|F\|_{L^q} \qquad \forall\ q > 1$$

we can give a proof which only rests on the use of Vitali's covering theorem and on the mean value property of harmonic functions

Open problems

- In the case

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- In the case of

$$\operatorname{div}\ (|Du|^{p-2}Du) = \operatorname{div}\ (|F|^{p-2}F)$$

the analog would write

$$p - 1 < q \qquad\qquad (\text{instead of } p \leq q)$$

that remains a very difficult open problem

Open problems

- Observe that the interval $q > p-1$ is the largest allowing for a distributional formulation of the equation

$$\text{div } (|Du|^{p-2}Du) = \text{div } (|F|^{p-2}F)$$

i.e.

$$\int \langle |Du|^{p-2}Du, D\varphi\rangle \, dx = \int \langle |F|^{p-2}F, D\varphi\rangle \, dx \qquad \forall\ \varphi \in C^\infty$$

makes sense

First answers

Theorem (Iwaniec & Sbordone, Crelle J. 93)

If u solves

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where ε does not depend on the solution

- Lewis (Comm. PDE 93) (with an alternative approach working for higher order equations)
- Kinnunen & Lewis (Duke Math. J. 2000, Ark. Math. 2002) in the case of systems

3 - Below the duality exponent - measure data

Assumptions

- **Consider the standard ellipticity assumptions**

$$\begin{cases} |a(z)| + |\partial a(z)||z| \leq L|z|^{p-1} \\ \nu|z|^{p-2}|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda\rangle \end{cases}$$

for brevity we shall confine ourselves to the case

$$p \geq 2$$

while optimal results are also available for the subquadratic case

Solvability

- Consider the problem

$$\begin{cases} -\text{div } a(Du) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $|\mu|(\Omega) < \infty$

- **The linear case** is a classical result of Littman & Stampacchia & Weinberger (Ann. SNS Pisa 1963)

SOLA (Boccardo & Gallouet, Dall'Aglio)

- We solve the approximating problems

$$\begin{cases} -\text{div } a(Du_k) = f_k \in L^\infty & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

$$f_k \to \mu$$

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- We solve the approximating problems

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$$f_k \to \mu$$

- and then $k \to \infty$

$$u_k \to u \qquad \text{strongly in } W^{1,p-1}$$

More definitions

- See the basic paper of Dal Maso, Murat, Orsina & Prignet (Ann. Scu. Norm. Pisa 99) for more on existence and definitions of solutions
- Kilpeläinen, Kuusi & Tuhola-Kujanpää (Ann. IHP 12) recently proved that all these definitions are equivalent in the case of positive measures

The "fundamental solution"

- The problem

$$\begin{cases} -\mathrm{div}(|Du|^{p-2}Du) = \delta & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- has a unique SOLA

$$G_p(x) \approx \begin{cases} |x|^{\frac{p-n}{p-1}} & \text{if } p \neq n \\ -\log|x| & \text{if } n = p \end{cases}$$

- that can be eventually used to test the optimality of the results

CZ theory for measure data problems: basic regularity

Theorem (Boccardo & Gallouët, JFA 89 - CPDE 92)

- *If μ is a measure then* $|Du|^{p-1} \in \mathcal{M}^{n/(n-1)}$
- *If $\mu \in L^q$ con*

$$1 < q < \frac{np}{np - n + p} = (p^*)'$$

then

$$|Du|^{p-1} \in L^{nq/(n-q)}$$

The case $p = n$ is due to Dolzmann & Hungerbühler & Müller, Crelle J. 00

Basic questions

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- The equation is formally a second order one, so: Can we expect differentiability of the gradient?
- On the other hand in the case

$$\triangle u = \mu \in L^1$$

in general we have

$$Du \notin W^{1,1}$$

Fractional derivatives

- **Fractional Sobolev spaces**
- **We say that $v \in W^{s,\gamma}$ with**

$$0 < s < 1 \qquad \gamma \geq 1$$

iff

$$\int\int \frac{|v(x) - v(y)|^\gamma}{|x - y|^{n+s\gamma}}\, dx\, dy < \infty$$

- **Intuitively**

$$\int\int \frac{|v(x) - v(y)|^\gamma}{|x - y|^{n+s\gamma}}\, dx\, dy \approx \int |D^s v(x)|^\gamma\, dx$$

CZ theory for measure data problems: maximal regularity

Theorem (Min., Ann. SNS Pisa 2007)

If u solves

$$-\operatorname{div} a(Du) = \mu \qquad \textit{with} \qquad p = 2$$

then

$$Du \in W^{1-\varepsilon,1} \qquad\qquad \forall \ \ \varepsilon > 0$$

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When $p > 2$

- **Forecasting....** We recall that

$$\frac{1}{|x|^{\beta}} \in W^{s,\gamma}(B) \Longleftrightarrow \beta < \frac{n}{\gamma} - s$$

We apply this fact to the fundamental solution

$$G_p \approx \frac{1}{|x|^{\frac{n-p}{p-1}}} \qquad |DG_p| \approx \frac{1}{|x|^{\frac{n-1}{p-1}}}$$

with the natural choice $\gamma = p - 1$, we obtain

$$s < \frac{1}{p-1}$$

therefore we would expect

$$Du \in W^{s,p-1} \qquad \forall \;\; s < \frac{1}{p-1}$$

Indeed....

Theorem (Min., Ann. SNS Pisa 2007)

If u solves

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Observe that since we are dealing with the case $p \geq 2$ then

$$\frac{1}{p-1} \leq 1$$

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Additional results are in Min., Math Ann. 2010 - The parabolic case has been treated by Baroni & Habermann (JDE 2012)

The result is optimal

- Fractional Sobolev embedding gives

$$W^{\sigma,q} \hookrightarrow L^{\frac{nq}{n-\sigma q}} \qquad \sigma q < n$$

- therefore assuming

$$Du \in W^{\frac{1}{p-1},p-1}$$

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that does not hold for the fundamental solution

4 - Linear and nonlinear potentials

Back to the classics

- **For the model case**

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

- **it hold**

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

and

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

The truncated Riesz potential

- **A version suited to bounded domains**

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho} \qquad \beta \in (0,n]$$

- **then**

$$\begin{aligned}\mathbf{I}^{\mu}_{\beta}(x,R) &\lesssim \int_{B_R(x)} \frac{d\mu(y)}{|x-y|^{n-\beta}} \\ &= I_\beta(\mu \llcorner B(x,R))(x) \leq I_\beta(\mu)(x)\end{aligned}$$

for all non-negative measures

Looking for nonlinear analogs

- Take nonlinear equations of the type

$$-\text{div }a(Du) = \mu$$

as before

- and for instance degenerate cases as

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Looking for nonlinear analogs

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  as before

- and for instance degenerate cases as

$$-\text{div } (|Du|^{p-2}Du) = \mu$$

- recalling that we are considering $p \geq 2$

New potentials are needed when $p \neq 2$

- Are the estimates

$$|u(x)| \lesssim I_2(|\mu|)(x) \qquad \text{e} \qquad |Du(x)| \lesssim I_1(|\mu|)(x)$$

valid for solutions to $-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$?

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would yield

$$u \equiv 0$$

Nonlinear Wolff potentials

- **The nonlinear Wolff potential** is defined by

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

with

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- when $p = 2$ it coincides with the Riesz

$$\mathbf{I}^{\mu}_{|\beta|}(x,R) := \int_0^R \frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n]$$

- **Nonlinear Wolff potentials** play a key role in nonlinear potential theory (similar to Riesz's in linear potential theory)

The first nonlinear potential estimate

Theorem (Kilpeläinen-Malý, Acta Math. 94)

If u solves

$$-\text{div }(|Du|^{p-2}Du) = \mu$$

then

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B(x,R)} |u|^{p-1}\,dy\right)^{1/(p-1)}$$

holds for a.e. x

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For $p = 2$ we are back to the Riesz potential $\mathbf{W}^{\mu}_{1,p} = \mathbf{I}^{\mu}_{2}$ - the above estimate is non-trivial already in this situation

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The first nonlinear potential estimate

- A new and important approach to the previous estimate has been given by Trudinger & Wang (Amer. J. Math. 2002). The result extends to the general subelliptic setting by their method

Optimal integrability properties of u now follow as a corollary

- **Indeed**

$$\mu \in L^q \Longrightarrow \mathbf{W}^{\mu}_{\beta,p} \in L^{\frac{nq(p-1)}{n-qp\beta}} \qquad q \in (1,n)$$

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- **This property follows by another pointwise estimate**

$$\int_0^\infty \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho} \lesssim I_\beta\left\{[I_\beta(|\mu|)]^{1/(p-1)}\right\}(x)$$

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- The quantity in the right-hand side is usually called Havin-Mazya potential

A first gradient potential estimate

Theorem (Min., JEMS 2011)

When $p = 2$, *if* u *solves*

$$-\operatorname{div} a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{I}_1^{|\mu|}(x, R) + \fint_{B(x,R)} |Du| \, dy$$

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holds for a.e. x

For solutions in $W^{1,1}(\mathbb{R}^N)$ we have

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

A gradient estimate with nonlinear potentials

Theorem (Duzaar & Min., Amer. J. Math. 2011)

If u solves

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \mu$$

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that actually means that

$$|Du(x)| \lesssim \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-1}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho} + \fint_{B(x,R)} |Du|\, dy$$

New viewpoint - Let's twist!!!

- Consider again
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$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$
- Being nonlinear in the gradient, the equation is still linear in $|Du|^{p-2}Du$ whose norm is $|Du|^{p-1}$
- Some brave thinking leads to conjecture the existence of a linear estimate for $|Du|^{p-1}$
- Also observe that a similar argument is not possible if we think of u

New viewpoint - Let's twist!!!

- Consider

$$-\text{div}\, v = \mu$$

with

$$v = |Du|^{p-2}Du$$

Indeed

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2012)

If u solves

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then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \left(\fint_{B(x,R)} |Du|\, dy\right)^{p-1}$$

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Indeed

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holds for a.e. x

The theorem still holds for general equations of the type $-\operatorname{div} a(Du) = \mu$

General operators

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2012)

If u solves

$$-\operatorname{div} a(x, Du) = \mu$$

and

$$x \mapsto a(x, \cdot) \text{ is Dini-continuous}$$

then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x, R) + \left(\fint_{B(x,R)} |Du| \, dy \right)^{p-1}$$

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The Dini continuity of coefficients is optimal

Same estimates for linear and nonlinear equations

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2012)

If u solves

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \mu \qquad \textit{in} \ \ \mathbb{R}^n$$

then

$$|Du(x)|^{p-1} \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

holds for a.e. x

- **For the model case equation** div $(|Du|^{p-2}Du) = \mu$ **the previous estimate implies for instance the local estimates included in almost all the papers devoted to the subject**
- **Moreover, the borderline cases which appeared as open problems in some of the above papers papers now follow as a corollary**

Lipschitz regularity and beyond

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2012)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

and

$$\lim_{R\to 0} \mathbf{I}_1^{|\mu|}(x,R) = 0 \text{ uniformly w.r.t. } x$$

then

Du is continuous

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then

$$Du \text{ is continuous}$$

There's no difference between Laplacean and p-Laplacean up to $C^{1,0}$-regularity

The general continuity criterion

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2012)

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then

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The general continuity criterion

Corollary

If u solves

$$-\operatorname{div} a(x, Du) = \mu$$

and

$$\mu \in L(n, 1)$$

that is

$$\int_0^\infty |\{x \in \Omega \, : \, |\mu(x)| > \lambda\}|^{1/n} d\lambda < \infty$$

then

Du is continuous

Developments

- Parabolic estimates for the evolutionary p-Laplacean equation

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \mu$$

  they use **intrinsic potentials** built following intrinsic geometries. This is joint work with **T. Kuusi**

- Suitable potential estimates for fully nonlinear equations

$$F(x, D^2u) = f$$

  using suitably modified potentials according to the ABP principle. This is joint work with **P. Daskalopoulos & T. Kuusi**

Fully nonlinear equations

Theorem (Daskalopoulos & Kuusi & Min., Preprint 2012)

Let u be an L^p-viscosity solution to

$$F(D^2 u) = \mu$$

with $n_e < p < n$. Then if $\mu \in L(n, 1)$ that is

$$\int_0^\infty |\{x \in \Omega \,:\, |\mu(x)| > \lambda\}|^{1/n} d\lambda < \infty$$

then Du is continuous

- This provides the sharp borderline case of the celebrated work of Caffarelli (Ann. Math. 89)
- More results are available for fully nonlinear, and concerning the sharp gradient integrability

Thank you – with a funny image I got from a PMS