Almost sure well-posedness for evolution equations

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Goal of the Talk

When deterministic statements about existence, uniqueness and stability of solutions to certain evolution equations are not easy to prove or not available all together, we turn to a more probabilistic point of view.

In this talk we will distinguish between equations that can be viewed as infinite dimension Hamiltonian systems, for example:

- Certain Schrödingier equations on compact manifolds
- Periodic KdV equations
- Certain wave equations on bounded domains

and those that do not enjoy this strong property.

We will start by

- Explaining what we mean with almost sure well-posedness
- Explaining why this approach improves in some sense the regularity of the problems at hand.

The General Set Up

Consider the initial value problem written in general form:

(GIVP) $$\begin{cases} u_t + P(D)u = F(u) & x \in M,\ t > 0 \\ u(x,0) = u_0(x), \end{cases}$$

where M is a manifold without boundaries, $P(D)$ is a certain differential operator, $F(u)$ is the nonlinear part of the equation, u_0 is the initial datum. Let us assume that u_0 belongs to a certain Banach space of functions X^s, with derivatives of order s.

Well-posedness (local) for us means that:

Definition

For any $u_0 \in X^s$ there exist $T > 0$ and a unique solution u to (GIVP) in $C([0,T], X^s)$ that is also stable in the appropriate topology.

Remark

If s is small, or if $F(u)$ is highly non linear, or if M is of high dimension or not flat, proving well-posedness could be very difficult and in some cases not even true. Counterexamples can be constructed!

Happy with Less!

One could think about introducing a probability measure μ in the space of initial data X^s that would give the following weaker version of well-posedness:

Definition

There exists $Y^s \subset X^s$, with $\mu(Y^s) = 1$ and such that for any $u_0 \in Y^s$ there exist $T > 0$ and a unique solution u to (GIVP) in t$C([0, T], X^s)$ that is also stable in the appropriate topology.

When this can be proved we will be talking about Almost Sure Well-posedness.

Remark

In what follows we will assume that $M = \mathbb{T}^d$, the torus of dimension d and that $X^s = H^s$, the usual L^2 based Sobolev space, but one can be much more general.

An Explicit Example of μ

Let $f \in H^s(\mathbb{T}^d)$ and let $(a_n)_{n\in\mathbb{Z}^d}$ be its Fourier coefficients. Let $I(\omega)_{n\in\mathbb{Z}^d}$ be independent identically distributed standard Gaussian or Bernulli random variables on a probability space $(\Omega, \mathcal{F}, P)$. We then defined the randomization of f to be

$$f^\omega(x) = \sum_{n\in\mathbb{Z}^d} a_n I_n(\omega) e^{ix\cdot n}.$$

The gain in using a randomization is in the fact that the randomized function gains regularity, as the following proposition shows:

Proposition

For $p \geq 2$

$$\|f^\omega\|_{L^p(\mathbb{T}^d)} \leq C\|a_n\|_{l^2} \sim \|f\|_{L^2(\mathbb{T}^d)}$$

almost surely.

Remark

One should at the same time remark, that the gain in regularity is only with respect to the exponent p **not** *s, the order of derivative, see for example N. Burq and N. Tzvetkov.*

Equations in Hamiltonian form

Consider the following equations with associated Hamiltonians:

The **Nonlinear Schrödinger equation in $\mathbb{T}^d$**

$$i\partial_t u + \Delta u + \lambda |u|^p u = 0 \quad \text{and} \quad H(u) = \frac{1}{2}\int |\nabla u|^2\,dx - \frac{\lambda}{p+2}\int |u|^{p+2}\,dx$$

for $p > 1$ and $\lambda = \pm 1$.

The **KdV equation in $\mathbb{T}$**

$$\partial_t u + \partial_{xxx} u + \lambda \partial_x (u^{k+1}) = 0 \quad \text{and} \quad H(u) = \frac{1}{2}\int |\partial_x u|^2\,dx - \frac{\lambda}{k+2}\int u^{k+2}\,dx$$

for $k \in \mathbb{N}$ and $\lambda = \pm 1$.

The **Derivative NLS equation in $\mathbb{T}$**

$$i\partial_t u + \partial_{xx} u + \lambda \partial_x (|u|^2 u) = 0$$

with Hamiltonian

$$H(u) = \frac{1}{2}\int |\partial_x u|^2\,dx + \frac{3}{4} Im \int u^2 \bar{u} \partial_x \bar{u}\,dx - \frac{\lambda}{4}\int u^6\,dx$$

for $\lambda = \pm 1$.

The **NLW equation in a bounded domain**

$$\partial_{tt} u - \Delta u + \lambda u^k = 0 \quad \text{and} \quad H(u) = \frac{1}{2}\int [|\partial_t u|^2 + |\nabla u|^2]\, dx - \frac{\lambda}{k+1}\int u^{k+1}\, dx$$

for $k \in \mathbb{N}$ and $\lambda = \pm 1$.

If one rewrites these equation for the Fourier coefficients (a_n) of u instead of u itself, then one obtains infinite dimension Hamiltonian systems.

For these systems one may hope to make sense of a very special kind of measure in the set of the initial data: the Gibbs measure and use it to show well-posedness in very low regularity regimes.

This method goes back to J. L. Lebowitz, H. A. Rose, and E. R. Speer and Zhidkov, then continued by J. Bourgain for NLS, KdV etc, and then more recently by N. Burq and N. Tzvetkov for the NLW, T. Oh for KdV type systems, A. Nahmod, T. Oh, L. Rey-Bellet, G.S for DNLS (see also L. Thomann and N. Tzvetkov).

The Gibbs measure: finite dimension

Hamilton's equations of motion have the antisymmetric form

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

the Hamiltonian $H(p,q)$ being conserved.

By defining $y := (q_1, \ldots, q_k, p_1, \ldots, p_k)^T \in \mathbb{R}^{2k}$ $(2k = d)$ we can rewrite the system in the compact form

$$\frac{dy}{dt} = J\nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

As a consequence of Liouville's Theorem the Lebesgue measure ν on $\mathbb{R}^{2k}$ is invariant under the Hamiltonian flow Φ_t:

$$\nu(\Phi_t(A)) = \nu(A)$$

for all measurable sets A.

A more interesting measure is the Gibbs measure. We have in fact:

Theorem (Invariance of Gibbs measures)

Assume that Φ_t is the flow generated by the Hamiltonian system above. Then the Gibbs measures defined as

$$d\mu := e^{-\beta H(p,q)} \prod_{i=1}^{d} dp_i \, dq_i$$

with $\beta > 0$, are invariant under the flow Φ_t.

The proof is trivial since from conservation of the Hamiltonian H the functions $e^{-\beta H(p,q)}$ remain constant, while, thanks to Liouville's Theorem the volume measure $d\nu = \prod_{i=1}^{d} dp_i \, dq_i$ remains invariant as well.

The Gibbs measure: infinite dimension

As mentioned before in more general terms, if we consider for example the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^4 u \\ u(0,x) = u_0(x), \;\text{ where }\; x \in \mathbb{T}. \end{cases}$$

with Hamiltonian

$$H(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx + \frac{1}{6}\int |u(t,x)|^6\,dx.$$

One can rewrite the Cauchy problem as

$$\dot{u} = i\frac{\partial H(u,\bar{u})}{\partial \bar{u}}$$

and if we think of u as the infinite dimension vector given by its Fourier coefficients $(\hat{u}(n))_{n\in\mathbb{Z}} = (a_n + ib_n)_{n\in\mathbb{Z}}$, then this becomes an infinite dimension Hamiltonian system for the vector $(a_n(t), b_n(t))_{n\in\mathbb{Z}}$.

Lebowitz, Rose and **Speer** considered the Gibbs measures *formally* given by

$$\text{“}d\mu = \exp\left(-\beta H(u)\right)\prod_{x\in\mathbb{T}} du(x)\text{”}$$

for $\beta > 0$ and showed that μ is a well-defined probability measure on $H^s(\mathbb{T})$ for any $s < \frac{1}{2}$.

The Guassian Measure

How does one make sense of the Gibbs measure introduced above? We need to go through the Gaussian measure.

For the example we are considering note that the quantity

$$H(u) + \frac{1}{2}\int |u|^2(x)\,dx$$

is conserved. Then the best way to make sense of the Gibbs measure μ is by writing it as

$$d\mu = \exp\left(-\frac{1}{6}\int |u|^6\,dx\right)\exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right)\prod_{x\in\mathbb{T}} du(x).$$

In this expression $d\rho = \exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right)\prod_{x\in\mathbb{T}} du(x)$ is the Gaussian measure and

$$\frac{d\mu}{d\rho} = \exp\left(-\frac{1}{6}\int |u|^6\,dx\right),$$

corresponding to the nonlinear term of the Hamiltonian, is understood as the Radon-Nikodym derivative of μ with respect to ρ.

On the other hand the Gaussian measure ρ is the weak limit of the finite dimensional Gaussian measures

$$d\rho_N = \exp\Big(-\frac{1}{2}\sum_{|n|\le N}(1+|n|^2)|\widehat{u}_n|^2\Big)\prod_{|n|\le N} da_n db_n.$$

Remark

The measure ρ_N above can be regarded as the induced probability measure on $\mathbb{R}^{2N+2}$ under the map

$$\omega \longmapsto \left\{\frac{I_n(\omega)}{\sqrt{1+|n|^2}}\right\}_{|n|\le N}.$$

where $I_n(\omega)$, $0 \le |n| \le N$, are independent standard complex Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ ($\widehat{u}_n = \frac{I_n}{\sqrt{1+|n|^2}}$). In a similar manner, we can view ρ as the induced probability measure under the map $\omega \mapsto \left\{\frac{I_n}{\sqrt{1+|n|^2}}\right\}_{n\in\mathbb{Z}}$.

Invariance of the Gibbs Measure and Almost Sure Global Well-posedness

Using the setting recalled above Bourgain proved

Theorem

Consider the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = \pm |u|^4 u \\ u(0,x) = u_0(x), \ \ \textit{where } x \in \mathbb{T}. \end{cases}$$

The Gibbs measure μ is well defined in $H^s, \ 0 \leq s < 1/2$ and there exists $\Omega \subset H^s$, such that $\mu(\Omega) = 1$, where the Cauchy problem is globally well-posed in Ω. Moreover μ is **invariant**.

Remark

To be precise, for the focusing case, one needs to impose the restriction that the mass (L^2 norm) is small.

Strength and Weaknesses of the Gibbs Measure

- Why is the use of the Gibbs measure more effective?
  - Because failure to show global existence with other methods (see Bourgain's high-low method or the I-method) might come from certain 'exceptional' initial data set, and the virtue of the Gibbs measure is that it does not see that exceptional set.
  - The invariance of the Gibbs measure, just like the usual conserved quantities, can be used to control the growth in time of those solutions in its support and extend the local in time solutions to global ones almost surely.
- What are the limitations of the Gibbs measure?
  - The difficulty in this approach lies in the actual construction of the associated Gibbs measure and in showing both its invariance under the flow and the almost sure global well-posedness.
  - In higher dimensions the Gibbs measure lives in very rough spaces where key estimates are not available.
  - If the equation is not Hamiltonian things are not clear.

Make Do with Less

N. Burq and N. Tzvetkov for certain supercritical NLW and J. Colliander and T. Oh for the 1D cubic NLS below L^2, abandoned the use of the Gibbs measure and only used the randomization of the initial data to obtain local, and then global, almost sure well-posedness.

The main idea here goes as follows: Consider again

$$\text{(GIVP)} \qquad \begin{cases} u_t + P(D)u = F(u) & x \in M, \ t > 0 \\ u(x,0) = u_0(x), \end{cases}$$

and assume that $u_0 \in X^s$, with s small.

- Randomize the initial datum as proposed above, call it u_0^ω.
- Assume v^ω is the solution of the associated linear problem with initial datum u_0^ω.
- Use the fact that v^ω has better L^p estimates than u_0 almost surely to show that $w = u - v^\omega$ solves a difference equation that lives in a smoother space than X^s. Obtain for w a *deterministic* local well-posedness.
- Use energy method (N. Burq and N. Tzvetkov) or high-low method of Bourgain (J. Colliander and T. Oh) to pass from local to global.

Some Recent Results

Following this line of work we would like to report two recent results

- For any fixed interval of time $[0, T]$:
  - The 2d periodic Navier-Stokes Cauchy problem is almost surely globally well-posed in $[0, T]$ for divergence free data in H^s, $s > -1/2$.
  - The 3d periodic Navier-Stokes Cauchy problem almost surely has weak solutions in $[0, T]$ for divergence free data in H^s, $s > -1/3$.

  (Joint work with A. Nahmod and N. Pavlovic.)

- The 3d periodic quintic NLS Cauchy problem is almost surely locally well-posed in H^s, $s > 1 - \sigma$, for some $\sigma > 0$. (Note that $s = 1$ is critical!)

  (Joint work with A. Nahmod.)

In the rest of the lecture will be devoted to the first result.

The Navier-Stokes Equations

Consider a viscous, homogenous, incompressible fluid with velocity $\vec{u}$ on $M = \mathbb{R}^d$ or $\mathbb{T}^d$, d=2, 3 and which is not subject to any external force. Then the initial value problem for the Navier-Stokes equations is given by

$$\text{(NSEp)} \qquad \begin{cases} \vec{u}_t + \vec{v} \cdot \nabla \vec{u} = -\nabla p + \nu \Delta \vec{u}; \quad x \in M, \ t > 0 \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(x, 0) = \vec{f}(x), \end{cases}$$

where $0 < \nu =$inverse Reynols number (non-dim. viscosity);
$\vec{u} : \mathbb{R}^+ \times M \to \mathbb{R}^d$, $p = p(x, t) \in \mathbb{R}$ and $\vec{f} : M \to \mathbb{R}^d$ is divergence free.

- For smooth solutions it is well known that the pressure term p can be eliminated via Leray-Hopf projections and view (NSEp) as an evolution equation of $\vec{u}$ alone[1],
- the mean of $\vec{u}$ is easily seen to be an invariant of the flow (conservation of momentum) so can reduce to the case of mean zero $\vec{f}$.

[1] Although understanding the pressure term might be important.

Then the incompressible Navier-Stokes equations (NSEp) (assume $\nu = 1$) can be expressed as

$$\text{(NSE)} \qquad \begin{cases} \vec{u}_t = \Delta \vec{u} - \mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{u}); \quad x \in \Omega, \quad t > 0 \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(x,0) = \vec{f}(x), \end{cases}$$

where $\mathbb{P}$ is the Leray-Hopf projection operator into divergence free vector fields given via

$$\mathbb{P}\vec{h} = \vec{h} - \nabla \frac{1}{\Delta}(\nabla \cdot \vec{h}) = (I + \vec{R} \otimes \vec{R})\vec{h}$$

($\vec{R}$ = Riesz transforms vector) and $\vec{f}$ is mean zero and divergence free.

By Duhamel's formula we have

$$\text{(NSEi)} \qquad \vec{u}(t) = e^{t\Delta}\vec{f} + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{u})\, ds$$

- In fact, under suitable general conditions on $\vec{u}$ the three formulations (NSEp), (NSE) and (NSEi) can be shown to be equivalent (weak solutions, mild solutions, integral solutions. Work by Leray, Browder, Kato, Lemarie, Furioli, Lemarie and Terraneo, and others.)

- Recall if the velocity vector field $\vec{u}(x,t)$ solves the Navier-Stokes equations in $\mathbb{R}^d$ or $\mathbb{T}^d$ then $\vec{u}_\lambda(x,t)$ with

$$\vec{u}_\lambda(x,t) = \lambda \vec{u}(\lambda x, \lambda^2 t),$$

is also a solution to the system (NSE) for the initial data

$$\vec{f}_\lambda = \lambda \vec{f}(\lambda x) \quad .$$

In particular,

$$\|\vec{f}_\lambda\|_{\dot{H}^{s_c}} = \|\vec{f}\|_{\dot{H}^{s_c}}, \qquad s_c = \frac{d}{2} - 1.$$

The spaces which are invariant under such a scaling are called critical spaces for Navier-Stokes. Examples:

$$\dot{H}^{\frac{d}{2}-1} \hookrightarrow L^d \hookrightarrow \dot{B}^{-1+\frac{d}{p}}_{p,\infty} \hookrightarrow BMO^{-1} \qquad (1 < p < \infty).$$

$v \in BMO^{-1}$ iff $\exists\ h^i \in BMO$ such that $v = \sum \partial_i h^i$ (Koch-Tataru)

- Classical solutions to the (NSE) satisfy the decay of energy which can be expressed as:

$$\|u(x,t)\|_{L^2}^2 + \int_0^t \|\nabla u(x,\tau)\|_{L^2}^2 \, d\tau = \|u(x,0)\|_{L^2}^2 .$$

- **When $d = 2$:** the energy $\|u(x,t)\|_{L^2}$, which is globally controlled, is exactly the scaling invariant $\dot{H}^{s_c} = L^2$-norm. In this case the equations are said to be *critical*. Classical global solutions have been known to exist; see Ladyzhenskaya (1969).
- **When $d = 3$:** the global well-posedness/regularity problem of (NSE) is a long standing open question!
  - The energy $\|u(x,t)\|_{L^2}$ is at the super-critical level with respect to the scaling invariant $\dot{H}^{\frac{1}{2}}$-norm, and hence the Navier-Stokes equations are said to be *super-critical*
  - The lack of a known bound for the $\dot{H}^{\frac{1}{2}}$ contributes in keeping the large data global well-posedness question for the initial value problem (NSE) still open.

Periodic Navier-Stokes Equation Below L^2

We consider the periodic Navier-Stokes problem (NSE)

$$\text{(NSE)} \qquad \begin{cases} \vec{u}_t = \Delta \vec{u} - \mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{u}); \quad x \in \mathbb{T}^d \quad t > 0 \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(x,0) = \vec{f}(x), \end{cases}$$

where $d = 2, 3$ and $\vec{f}$ is divergence free and mean zero and $\mathbb{P}$ is the Leray projection into divergence free vector fields.

- We address the question of long time existence of weak solutions for super-critical randomized *large* initial data both in $d = 2$, 3.
- For $d = 2$ we address uniqueness as well.

Navier-Stokes: A Road Map of the Argument

- We start with an initial data $\vec{f} \in H^{-\alpha}$, $\alpha > 0$, hence supercritical. Assume $\{\vec{a}_n\}$ are the Fourier coefficients of $\vec{f}$.
- Randomizing $\vec{f}$ means that we replace $\{\vec{a}_n\}$ by $\{l_n(\omega)\vec{a}_n\}$, where $\{l_n(\omega)\}$ are independent random variables, and we take its Fourier series $\vec{f}^{\omega}$ as the new randomized initial data.
- We seek a solution to the initial value problem (NSE) in the form $\vec{u} = e^{t\Delta}\vec{f}^{\omega} + \vec{w}$ and identify the difference equation that $\vec{w}$ should satisfy.
- The heat flow of the randomized data gives almost surely improved L^p bounds. These bounds yield improved nonlinear estimates arising in the analysis of the difference equation for $\vec{w}$ almost surely.
- We revisit the proof of equivalence between the initial value problem for the difference equation and the integral formulation of it in our context.

- We prove a priori energy estimates for $\vec{w}$. The integral equation formulation is used near time zero and the other one away from zero.
- A construction of a global weak solution to the difference equation via a Galerkin type method is thus possible.
- We prove uniqueness of weak solutions when $d = 2$. Our proof is done 'from scratch' for the difference equation (in spirit of Ladyzhenskaya-Prodi-Serrin condition).
- Put all ingredients together to conclude.

Remark

We should immediately notice that although in our paper we use improved properties for $e^{t\Delta}\vec{f}^{\omega}$, one can show that already $\vec{f}^{\omega}$ belongs to certain **critical** *Besov spaces for which Gallagher-Planchon already proved in 2d global well-posedness. On the other hand while their proof is based on a combination of the high-low argument of Bourgain and the H. Kock-Tataru small BMO^{-1} data result, ours is much more self contained and gives more precise energy estimate. Moreover our existence result extends to 3d, as mentioned.*

Now the Details

We start by recolling that here we are dealing with divergence free initial data, so we need to pick a randomization that maintains this property:

Definition [Diagonal randomization]

Let $(l_n(\omega))_{n\in\mathbb{Z}^d}$ be a sequence of of real, independent, random variables on a probability space (Ω, A, p) For $\vec{f} \in (H^s(\mathbb{T}^d))^d$, let (a_n^i), $i = 1, 2, \dots, d$, be its Fourier coefficients. We introduce the map from (Ω, A) to $(H^s(\mathbb{T}^d))^d$ equipped with the Borel sigma algebra, defined by

$$\text{(DR)} \qquad \omega \longrightarrow \vec{f}^{\omega}, \qquad \vec{f}^{\omega}(x) = \left(\sum_{n\in\mathbb{Z}^d} l_n(\omega) a_n^1 e_n(x), \dots, \sum_{n\in\mathbb{Z}^d} l_n(\omega) a_n^d e_n(x) \right),$$

where $e_n(x) = e^{in\cdot x}$ and call such a map randomization.

Remarks

- The map (DR) is measurable and $\vec{f}^\omega \in L^2(\Omega; (H^s(\mathbb{T}^d))^d)$, is an $(H^s(\mathbb{T}^d))^d$-valued random variable.
- The diagonal randomization defined in (DR) commutes with the Leray projection $\mathbb{P}$.
- No H^s regularization $\|\vec{f}^\omega\|_{H^s} \sim \|\vec{f}\|_{H^s}$ (Burq-Tzvetkov).
- **But randomization gives improved L^p estimates (almost surely).**

Main Results

Theorem [Existence and Uniqueness in 2D]

Fix $T > 0$, $0 < \alpha < \frac{1}{2}$ and let $\vec{f} \in (H^{-\alpha}(\mathbb{T}^2))^2$, $\nabla \cdot \vec{f} = 0$ and of mean zero. Then there exists a set $\Sigma \subset \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (NSE) with datum $\vec{f}^\omega$ has a unique global weak solution $\vec{u}$ of the form

$$\vec{u} \,=\, \vec{u}_{\vec{f}^\omega} \,+\, \vec{w}$$

where $\vec{u}_{\vec{f}^\omega} = e^{t\Delta}\vec{f}^\omega$ and $\vec{w} \in L^\infty([0,T];(L^2(\mathbb{T}^2))^2) \cap L^2([0,T];(\dot{H}^1(\mathbb{T}^2))^2)$.

Theorem [Existence in 3D]

Fix $T > 0$, $0 < \alpha < \frac{1}{3}$ and let $\vec{f} \in (H^{-\alpha}(\mathbb{T}^3))^3$, $\nabla \cdot \vec{f} = 0$, and of mean zero. Then there exists a set $\Sigma \subset \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (NSE) with datum $\vec{f}^\omega$ has a global weak solution $\vec{u}$ of the form

$$\vec{u} \,=\, \vec{u}_{\vec{f}^\omega} \,+\, \vec{w},$$

where $\vec{u}_{\vec{f}^\omega} = e^{t\Delta}\vec{f}^\omega$ and $\vec{w} \in L^\infty([0,T];(L^2(\mathbb{T}^3))^3) \cap L^2([0,T];(\dot{H}^1(\mathbb{T}^3))^3)$.

Some Previous Results:

This approach has been already applied in the context of the Navier-Stokes to obtain:

- **Local in time** solutions to the corresponding integral equation for randomized initial data in $L^2(\mathbb{T}^3)$ by Zhang and Fang (2011) and by Deng and Cui (2011). Also **global in time** solutions to the corresponding integral equation for randomized **small initial data**.
- Deng and Cui (2011) obtained local in time solutions to the corresponding integral equation for randomized initial data in $H^s(\mathbb{T}^d)$, for $d = 2, 3$ with $-1 < s < 0$.

Free Evolution of the Randomized Data

Deterministic estimates.

For $0 < \alpha < 1$, $k \geq 0$ integer and $\vec{u}_{\vec{f}^\omega} = e^{t\Delta}\vec{f}^\omega$, $\vec{f}^\omega \in (H^{-\alpha}(\mathbb{T}^d))^d$, we have:

$$\begin{aligned} \|\nabla^k \vec{u}_{\vec{f}^\omega}(\cdot,t)\|_{L^2_x} &\lesssim (1 + t^{-\frac{\alpha+k}{2}})\, \|\vec{f}\|_{H^{-\alpha}}. \\ \|\nabla^k \vec{u}_{\vec{f}^\omega}\|_{L^\infty_x} &\lesssim \left(\max\{t^{-1}, t^{-(k+\alpha+\frac{d}{2})}\}\right)^{\frac{1}{2}}\, \|\vec{f}\|_{H^{-\alpha}}. \end{aligned}$$

Probabilistic estimates.

Let $T > 0$ and $\alpha \geq 0$. Let $r \geq p \geq q \geq 2$, $\sigma \geq 0$ and $\gamma \in \mathbb{R}$ be such that $(\sigma + \alpha - 2\gamma)q < 2$. Then there exists $C_T > 0$ such that for every $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d$

$$\|t^\gamma (-\Delta)^{\frac{\sigma}{2}} e^{t\Delta}\vec{f}^\omega\|_{L^r(\Omega; L^q([0,T]; L^p_x))} \leq C_T\, \|\vec{f}\|_{H^{-\alpha}},$$

where C_T may depend also on p, q, r, σ, γ and α.

Probabilistic estimates (cont.)

Moreover, if we set

$$E_{\lambda,T,\vec{f},\sigma,p} = \{\omega \in \Omega \,:\, \|t^{\gamma}(-\Delta)^{\frac{\sigma}{2}} e^{t\Delta}\vec{f}^{\omega}\|_{L^q([0,T];L^p_x)} \geq \lambda\},$$

then there exists $c_1, c_2 > 0$ such that for every $\lambda > 0$ and for every $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d$

$$P(E_{\lambda,T,\vec{f},\sigma,p}) \leq c_1 \exp\left[-c_2 \frac{\lambda^2}{C_T\|\vec{f}\|^2_{H^{-\alpha}}}\right].$$

Difference Equation. Equivalent Formulations

Let

$$\begin{aligned} H &= \text{the closure of } \{\vec{f} \in (C^\infty(\mathbb{T}^d))^d \,|\, \nabla \cdot \vec{f} = 0\} \text{ in } (L^2(\mathbb{T}^d))^d, \\ V &= \text{the closure of } \{\vec{f} \in (C^\infty(\mathbb{T}^d))^d \,|\, \nabla \cdot \vec{f} = 0\} \text{ in } (\dot{H}^1(\mathbb{T}^d))^d, \\ V' &= \text{ the dual of } V. \end{aligned}$$

and recall

$$\vec{u} - \vec{u}_{\vec{f}^\omega} =: \vec{w},$$

We consider two formulations of the initial value problem for the difference equation that $\vec{w}$ solves and re-prove in our context an equivalence lemma, which is similar to the version for the Navier-Stokes equations themselves (Lemarie, Furioli-Lemarie, Terraneo).

The Equivalence Lemma

Let $T > 0$. Assume that $\nabla \cdot \vec{g} = 0$, $\|\vec{g}(x,t)\|_{L^2} \lesssim (1 + \frac{1}{t^{\frac{\alpha}{2}}})$ and

$$\begin{cases} \|\vec{g}\|_{L^4([0,T],L^4_x)} \leq C, & \text{if } d = 2 \\ \|\vec{g}\|_{L^6([0,T],L^6_x)} \leq C, & \text{if } d = 3, \end{cases}$$

for some $C > 0$. Then the following statements are equivalent.

(DE) $\vec{w}$ is a weak solution to the initial value problem

$$\begin{cases} \partial_t \vec{w} = \Delta \vec{w} - \mathbb{P}\nabla(\vec{w} \otimes \vec{w}) + c_1[\mathbb{P}\nabla(\vec{w} \otimes \vec{g}) + \mathbb{P}\nabla(\vec{g} \otimes \vec{w})] + c_2\mathbb{P}\nabla(\vec{g} \otimes \vec{g}) \\ \nabla \cdot \vec{w} = 0, \\ \vec{w}(x,0) = 0. \end{cases}$$

(IE) The function $\vec{w} \in L^\infty((0,T); H) \cap L^2((0,T), V)$, solves

$$\vec{w}(t) = -\int_0^t e^{(t-s)\Delta} \nabla \vec{F}(x,s)\, ds, \qquad \text{where}$$

$$\vec{F}(x,s) = -\mathbb{P}(\vec{w} \otimes \vec{w}) + c_1[\mathbb{P}(\vec{w} \otimes \vec{g}) + \mathbb{P}(\vec{g} \otimes \vec{w})] + c_2\mathbb{P}(\vec{g} \otimes \vec{g}).$$

The Importance of the Equivalence Lemma

The Equivalence Lemma stated above is fundamental in our argument since it will be used heavily in proving:

- The Energy Estimate for $\vec{w}$. Near zero, where g is singular, we use a continuity augment while away from zero we use the usual argument for NS.
- The Proof of the existence of weak solutions via the energy estimate for $\vec{w}$ in conjunction with Galerkin type skeme.

See also the work by T. Tao (07').

Energy Estimates for the Difference Equation

$$E(\vec{w})(t) = \|\vec{w}(t)\|_{L^2}^2 + c\int_0^t \int_{\mathbb{T}^d} |\nabla \otimes \vec{w}|^2 \, dx \, ds$$

Theorem

Let $T > 0$, $\lambda > 0$, $\gamma < 0$, and $\alpha > 0$ be given. Let $\vec{g}$ be s.t. $\nabla \cdot \vec{g} = 0$ and

$$\|\vec{g}(x,t)\|_{L^2} \lesssim (1 + \frac{1}{t^{\frac{\alpha}{2}}}), \quad \|\nabla^k \vec{g}(x,t)\|_{L^\infty} \lesssim \left(\max\{t^{-1}, t^{-(k+\alpha+\frac{d}{2})}\}\right)^{\frac{1}{2}} \; k = 0, 1;$$

$$\begin{cases} \|t^\gamma \vec{g}\|_{L^4([0,T];L^4_x)} \leq \lambda, & \text{if } d = 2 \\ \|t^\gamma \vec{g}\|_{L^6([0,T];L^6_x)} \leq \lambda, & \text{if } d = 3. \end{cases}$$

Let $\vec{w} \in L^\infty((0,T);H) \cap L^2((0,T);V)$ be a solution to (DE). Then,

$$E(\vec{w})(t) \lesssim C(T, \lambda, \alpha), \qquad \text{for all } t \in [0, T].$$

$$\|\frac{d}{dt}\vec{w}\|_{L^p_t H^{-1}_x} \leq C(T, \lambda, \alpha),$$

where $p = 2$, if $d = 2$ and $p = \frac{4}{3}$, if $d = 3$.

Weak Solutions for the Difference Equation

Theorem

Let $T > 0$, $\lambda > 0$, $\gamma < 0$ and $\alpha > 0$ be given. Assume that the function $\vec{g}$ satisfies $\nabla \cdot \vec{g} = 0$ and

$$\|\vec{g}(x,t)\|_{L^2} \lesssim \left(1 + \frac{1}{t^{\frac{\alpha}{2}}}\right)$$

$$\|\nabla^k P_M \vec{g}(x,t)\|_{L^\infty} \lesssim \left(\max\{t^{-1}, t^{-(k+\alpha+\frac{d}{2})}\}\right)^{\frac{1}{2}} \quad \text{for } k = 0, 1.$$

Furthermore, assume that we have:

$$\begin{cases} \|t^\gamma \vec{g}\|_{L^4_{x;t\in[0,T]}} \leq \lambda, & \text{if } d = 2 \\ \|t^\gamma \vec{g}\|_{L^6_{x;t\in[0,T]}} \leq \lambda, & \text{if } d = 3. \end{cases}$$

Then there exists a weak solution $\vec{w}$ for the initial value problem (DE).

Proof of Main Theorems: Gathering all the Pieces

We find solutions $\vec{u}$ to (NSE) by writing

$$\vec{u} = \vec{u}^{\omega}_{\vec{f}} + \vec{w}$$

where we recall that $\vec{u}^{\omega}_{\vec{f}}$ is the solution to the linear problem with initial datum $\vec{f}^{\omega}$ and $\vec{w}$ is a solution to (DE) with $\vec{g} = \vec{u}^{\omega}_{\vec{f}}$.

- $\vec{u}$ is a weak solution for (NSE) if and only if $\vec{w}$ is a weak solution for (DE). We also remark that uniqueness of weak solutions to (DE) is equivalent to uniqueness of weak solutions (NSE).
- The proof of the existence of weak solutions is the same for both $d = 2$ and $d = 3$ and it is a consequence of the existence theorem above.

Let $\gamma < 0$ be such that

$$0 < \alpha < \begin{cases} \frac{1}{2} + 2\gamma, & \text{if } d = 2 \\ \frac{1}{3} + 2\gamma, & \text{if } d = 3. \end{cases}$$

By the probabilistic estimates with $\sigma = 0$, $p = q = 4$ when $d = 2$, and $p = q = 6$ when $d = 3$ we have that given $\lambda > 0$, if we define the set

$$E_\lambda := E_{\lambda,\alpha,\vec{f},\gamma,T} = \{\omega \in \Omega \ / \ \|t^\gamma \vec{u}^\omega_{\vec{f}}\|_{L^p_{[0,T],x}} > \lambda\},$$

there exist $C_1, C_2 > 0$ such that

$$P(E_\lambda) \leq C_1 \exp\left[-C_2 \left(\frac{\lambda}{C_T \|\vec{f}\|_{H^{-\alpha}}}\right)^2\right].$$

Now, let $\lambda_j = 2^j$, $j \geq 0$ and define $E_j = E_{\lambda_j}$. Note $E_{j+1} \subset E_j$. Let

$$\Sigma := \cup E_j^c \subset \Omega.$$

Then

$$1 \geq P(\Sigma) = 1 - \lim_{j\to\infty} P(E_j) \geq 1 - \lim_{j\to\infty} \exp\left[-C_2 \left(\frac{2^j}{C_T \|\vec{f}\|_{H^{-\alpha}}}\right)^2\right] = 1.$$

Final Step:

Our goal is now to show that for a fixed divergence free vector field $\vec{f} \in (H^{-\alpha}(\mathbb{T}^d))^d$ and for any $\omega \in \Sigma$, if we define $\vec{g} = \vec{u}^{\omega}_{\vec{f}}$, the initial value problem (DE) has a global weak solution. In fact given $\omega \in \Sigma$, there exists j such that $\omega \in E_j^c$. In particular we then have

$$\|t^{\gamma}\, \vec{g}\|_{L^p_{x,T}} \leq \lambda_j.$$

Hence assumptions on $\vec{g}$ in the previous theorems are satisfied. This concludes the proof.