Impulsive Gravitational Waves

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Introduction: Initial value problem in general relativity

Consider 3+1 dimensional Lorentzian manifold $(\mathcal{M}, g)$ satisfying the vacuum Einstein equations:

$$Ric_{\mu\nu}(g) = 0.$$

The simplest solution is given by $(\mathbb{R}^4, m)$ where

$$m = -2dud\underline{u} + dX^2 + dY^2.$$

Given a three dimensional Riemannian manifold with the induced first and second fundamental forms $(\bar{\mathcal{M}}, \bar{g}, \bar{k})$ satisfying the constraint equations, a unique local solution to the Einstein vacuum equations exists (Choquet-Bruhat).

The lowest reguarity known for a local solution with general data is given by the recently resolved L^2 curvature conjecture (Klainerman-Rodnianski-Szeftel).

Introduction: Initial value problem in general relativity

Alternatively, the Einstein equations can be solved by giving characteristic initial data on two intersecting null hypersurfaces (Rendall):

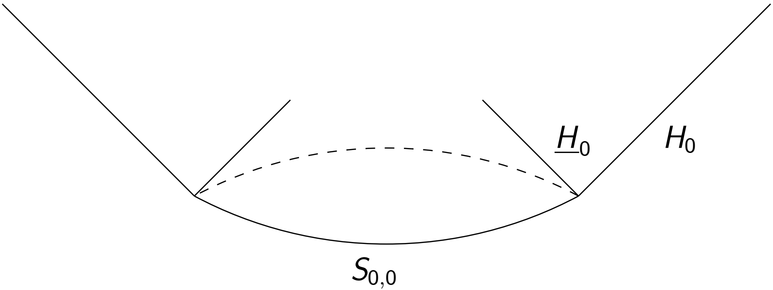

Figure : Setup of the characteristic initial value problem

Introduction: Penrose solution

Penrose (1972) discovered an explicit solution to the vacuum Einstein equations whose metric is given by

$$g = -2du d\underline{u} + (1 - \underline{u}\Theta(\underline{u}))dX^2 + (1 + \underline{u}\Theta(\underline{u}))dY^2,$$

where Θ is the Heaviside step function.
The Riemann curvature tensor (specifically, the only non-trivial components $R(\partial_{\underline{u}}, \partial_X, \partial_{\underline{u}}, \partial_X)$ and $R(\partial_{\underline{u}}, \partial_Y, \partial_{\underline{u}}, \partial_Y)$) is a delta function on the null hypersurface $\underline{u} = 0$. This was interpreted as an impulsive gravitational wave.

Introduction: Khan-Penrose solution

Khan-Penrose later discovered an explicit solution to the vacuum Einstein equations representing the interaction of two plane impulsive gravitational waves.

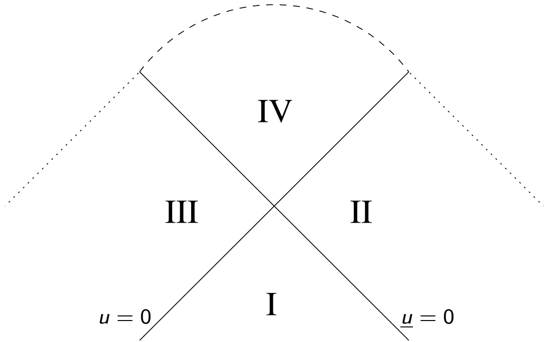

Figure : Schematic diagram of the Khan-Penrose solution

Beyond plane symmetry

Besides only being explicit special solutions, the Penrose and Khan-Penrose solutions are constructed in plane symmetry. Such an impulsive gravitational wave can only be thought of as an idealization that the source of the gravitational wave is at an infinite distance. Moreover, plane symmetry also assumes that the gravitational wave has infinite extent and automatically imposes the assumption of non-asymptotic flatness.

Characteristic initial value problem

We study the problem of impulsive gravitational waves viewed in the context of the characteristic initial value problem.
The freely prescribable data on each of the null hypersurface is the traceless part of the second fundamental form $\hat{\chi}$ and $\underline{\hat{\chi}}$. We prescribe data such that

- $\hat{\chi}$ has a jump discontinuity across an embedded 2-sphere $S_{0,\underline{u}_s}$ but smooth everywhere else
- $\underline{\hat{\chi}}$ is smooth (without any smallness assumptions)

Notice that $\nabla_{\partial_{\underline{u}}}\hat{\chi} = -\alpha + \ldots$, thus the curvature component $\alpha_{AB} = R(\partial_{\underline{u}}, e_A, \partial_{\underline{u}}, e_B)$ has a delta singularity.

Local propagation of impulsive gravitational wave

Theorem (L.-Rodnianski)

Given the initial data as above, a unique local solution to the vacuum Einstein equations exists. Moreover, the curvature has a delta singularity supported on the incoming null hypersurface emanating from the initial singularity and the spacetime is smooth away from this hypersurface.

Local propagation of impulsive gravitational wave II

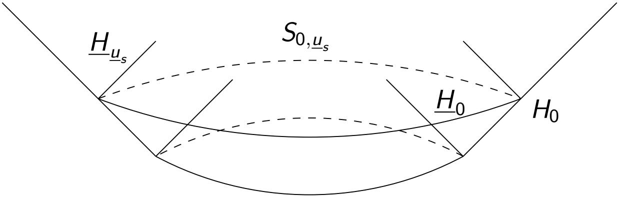

Figure : Local propagation of impulsive gravitational wave

Some remarks on the theorem

- The Riemann curvature tensor is not in L^2 and in terms of differentiability is one derivative weaker than the L^2 curvature conjecture. The metric is only Lipschitz.
- Nevertheless, the Einstein equations can be understood in L^2.
- In fact, we can show that for a sequence of smooth initial data approaching the given singular initial data, the solutions to the vacuum Einstein equations are smooth in a common domain of existence and approach the solution given by the theorem.
- Uniqueness can also be understood in the sense of limit, i.e., for any sequence of smooth data approaching the given singular initial data, the smooth solutions approach the unique solution given by the theorem.

The Einstein vacuum equations reformulated

Using the Einstein vacuum equations $R_{\mu\nu} = 0$ and the second Bianchi equations, we get a nonlinear wave equation for the Riemann curvature tensor

$$\Box R = R * R.$$

- Since the Riemann curvature tensor has a delta singularity in the data, we need to exploit the structure of the vacuum Einstein equations to make sense of the right hand side of this equation.
- The wave character of the equations suggests the use of L^2 based estimates. However, the curvature in the initial data does not belong to L^2!

Double null foliation

In order to capture the structure of the vacuum Einstein equations associated to null hypersurfaces, we use the double null foliation (previously used by Klainerman-Nicolo, Christodoulou, Klainerman-Rodnianski). Foliate the spacetime with incoming and outgoing null hypersurfaces. Define a null frame $\{e_1, e_2, e_3, e_4\}$ such that e_3 and e_4 are null and e_1 and e_2 are tangent to the spheres of intersection of the null hypersurfaces. Define the Ricci coefficients with respect to the null frame

$$\Gamma = g(D_{e_\mu} e_\nu, e_\delta),$$

and decompose the Riemann curvature tensor with respect to the null frame

$$\Psi = R(e_\mu, e_\nu, e_\delta, e_\sigma).$$

Energy estimates and transport equations

To prove existence and uniqueness, we need to prove a priori estimates. The basic strategy is to combine an L^2 energy estimates for the curvature tensor and transport equation estimates for the Ricci coefficients.

$$\int_{H_u} \Psi^2 + \int_{\underline{H}_{\underline{u}}} \Psi^2 \leq \int_{H_0} \Psi^2 + \int_{\underline{H}_0} \Psi^2 + \int_0^{\underline{u}} \int_0^{u} \int_{S_{u',\underline{u}'}} \Gamma \Psi \Psi \, du' d\underline{u}'.$$

In order to estimate the nonlinear error terms, we need to be able to estimate Γ by Ψ. This can be achieved by integrating the null structure equations

$$\nabla_3 \Gamma = \Psi + \Gamma\Gamma, \quad \nabla_4 \Gamma = \Psi + \Gamma\Gamma.$$

Renormalized energy estimates

There are various obvious difficulties in carrying out the above argument:

- Not all curvature components are defined in L^2! Even for the initial data, α is only defined as a measure. The hope, therefore, is to only control the other curvature components in L^2
- Need to control higher *angular* derivatives
- Need to show that this is sufficient to control the geometry of the spacetime
- Even when only estimating the other curvature components, α might come up in the error terms

Renormalized energy estimates II

It turns out that it is more suitable to estimate some renormalized version of the curvature components instead of the curvature components themselves, i.e., we subtract off an L^∞ correction to the curvature components such that the error term becomes bounded.

In order to prove L^2 estimates for the renormalized curvature components, the standard framework via the Bel Robinson tensor is ill-suited. Instead, we prove the L^2 estimates directly from the Bianchi equations (cf. Holzegel)

Renormalized energy estimates III

The Einstein equations together with second Bianchi equations can be written as

$$D^\mu R_{\mu\nu\delta\sigma} = 0$$

Decomposed with respect to the null frame, we have the following equations for $\beta_A = R(e_3, e_4, e_4, e_A)$ and $\rho = R(e_3, e_4, e_3, e_4)$:

$$\nabla_4 \rho = \text{div } \beta - \frac{1}{2}\underline{\hat{\chi}} \cdot \alpha + \Gamma\Psi$$

$$\nabla_3 \beta = \nabla \rho + \Gamma\Psi$$

Define

$$\check{\rho} = \rho - \frac{1}{2}\hat{\chi} \cdot \underline{\hat{\chi}},$$

and using the equation

$$\nabla_4 \hat{\chi} = -\alpha + \Gamma\Gamma,$$

we can re-write the equations without the curvature component α!

Existence and uniqueness

Since we are working with very rough solutions to the vacuum Einstein equations, a priori estimates do not automatically imply existence and uniqueness of the solutions. In particular, when proving the a priori estimates, the structure of the equations have been used and they may not be preserved when taking the difference of two spacetimes.

We therefore need to consider the difference of two spacetimes in a way that is adapted to the null geometry. It can in fact be shown that the difference of the geometric quantities can be controlled without any knowledge of the α component of either of the spacetime.

We have thus proved a theorem on existence and uniqueness of solutions to vacuum Einstein equations for a larger class of initial data!

Propagation of singularity and regularity

Once we have the existence and uniqueness of solutions, we can make use of the fact that α initially is a delta function across an embedded 2-sphere to recover more regularity of the spacetime:

- To show that α remains a delta measure, we need to show an L^1 type estimate. We pass to an approximate solution, which allows us to use the previously forbidden L^2 estimates for α. In fact, we can prove an L^2 estimate in a localized region of $\underline{u}$:

$$\int_{\{|\underline{u}-\underline{u}_s|\sim\delta\}} |\alpha| \lesssim \delta^{\frac{1}{2}}\left(\int_{\{|\underline{u}-\underline{u}_s|\sim\delta\}} |\alpha|^2\right)^{\frac{1}{2}} \lesssim 1$$

- To show that the spacetime is smooth beyond the impulsive gravitational wave, we integrate the Bianchi equations directly from the initial data using the null structure of the equations

$$\nabla_3\alpha = \nabla\beta + \Gamma\Psi.$$

  This estimate results in a loss of derivative, but we can use the bounds already obtained in the existence theorem.

Interaction of two impulsive gravitational waves

We now move on to study the interaction of two impulsive gravitational waves. Prescribe data such that *both* $\hat{\chi}$ and $\underline{\hat{\chi}}$ have a jump discontinuity across an embedded 2-sphere but are otherwise smooth. Surprisingly, even when the impulsive gravitational waves have curved wavefronts, no new singularities arise after their interaction.

Theorem (L.-Rodnianski)

With the given initial data, a unique local solution to the vacuum Einstein equations exists. Moreover, the curvature tensor of the solution has a delta singularity across the null hypersurface emanating from each of the singular 2-sphere and the solution is smooth away from the union of these two null hypersurfaces.

Interaction of two impulsive gravitational waves II

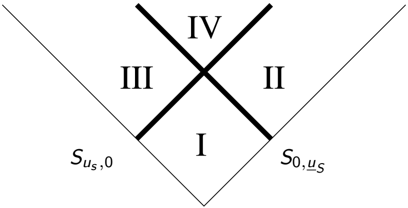

Figure : Interaction of two impulsive gravitational waves

Formation of trapped surface

An unexpected consequence of our approach to study impulsive gravitational waves is an improved theorem on formation of trapped surfaces.

Recall that a trapped surface is a two surface such that $\mathrm{tr}\chi < 0$ and $\mathrm{tr}\underline{\chi} < 0$. The classical theorem of Hawking and Penrose guarantees that if a vacuum spacetime has a trapped surface, the maximal Cauchy development is geodesically incomplete.

A question that is left unanswered by the Hawking-Penrose Theorem is whether a trapped surface can form dynamically. By the stability of Minkowski space proved by Christodoulou-Klainerman, in order to ensure that a trapped surface is formed, certain geometric quantities are large. This leads to the problem of constructing a semi-global large data solution to the vacuum Einstein equations.

Formation of trapped surface

This problem was solved by Christodoulou (2008), who showed that trapped surface can be formed dynamically. The novel idea was to introduce a small length scale δ and to impose largeness only in some components of curvature in terms of δ on H_0. On $\underline{H}_0$, the data is assumed to be trivial (Minkowskian) and the smallness can offset the largeness in the proof of the a priori estimates.

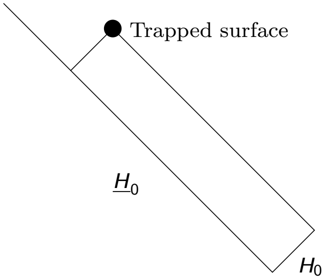

Figure : Formation of trapped surface

Formation of trapped surface

In applications, it is desirable to remove the assumption of the trivial data on $\underline{H}_0$. We study the case such that the data on $\underline{H}_0$ is not necessarily small, but satisfying some regularity assumptions. To achieve this, we prove estimates in a weaker norm such that the geometric quantities are no longer large with respect to δ. The large data of Christodoulou is of size

$$||\hat{\chi}||_{L^\infty} \sim \delta^{-\frac{1}{2}}, \quad ||\alpha||_{L^\infty} \sim \delta^{-\frac{3}{2}}.$$

- Using the renormalized energy estimates, we can control the spacetime without information of α
- Bounds for $||\hat{\chi}||_{L^\infty}$ can be replaced by bounds for $||\hat{\chi}||_{L^2}$ which is of size 1

Semi-global existence near one null hypersurface I

Besides proving estimates only using weaker control of the geometric quantities, we need to obtain the bounds in a region that is such that the $\underline{u}$ distance is a small number δ while the u distance is arbitrarily large. The challenge is that certain error terms in the estimates have to be integrated over the large u interval.

We can already ask whether we have existence and uniqueness of solutions to the vacuum Einstein equations even when the initial data is smooth.

Theorem (L.)

Given any smooth data on $H_0 \cap \{\underline{u} \leq 1\}$ and $\underline{H}_0 \cap \{u \leq 1\}$, there exists δ such that a unique solution to the vacuum Einstein equations in $\underline{u} \leq \delta$, $u \leq 1$.

Semi-global existence near one null hypersurface II

Once again, there is a null structure in the Einstein equations! For example, in the tranport equations for the Ricci coefficient:

$$\nabla_3\Gamma_3 = \Psi + \Gamma_3\Gamma_4, \quad \nabla_4\Gamma_4 = \Psi + \Gamma_3\Gamma_4.$$

In other words, we can first estimate the "good" components Γ_3 exploiting the small length scale, and then notice that the equations for the "bad" components Γ_4 are in fact linear in Γ_4! Additionally, this can be used together with the renormalized energy for the curvature and the L^2 control for the Ricci coefficients. This allows us to consider data that are smooth on $\underline{H}_0$ while on H_0

$$||\hat{\chi}||_{L^\infty} \sim \delta^{-\frac{1}{2}}, \quad ||\alpha||_{L^\infty} \sim \delta^{-\frac{3}{2}}.$$

Formation of trapped surface

Theorem (L.-Rodnianski)

Assume

$$\left(tr\chi(u=0,\vartheta)+\int_0^{u_*}\exp(\frac{1}{2}\int_0^{u'}tr\underline{\chi}(u'',\vartheta)du'')(-2K+2div\,\zeta+2|\zeta|^2)(u',\vartheta)du'\right)$$
$$<\exp(-\frac{1}{2}\int_0^{u_*}tr\underline{\chi}(u',\vartheta)du')tr\chi(u=0,\vartheta)\quad \textit{for all }\vartheta$$

holds on $\underline{H}_0$. *Then there exists an open set of initial data on* H_0 *that is free of trapped surface and such that a trapped surface is formed in its casual future.*

Thank you!