Calculus and heat flow in metric measure spaces and spaces with Riemannian curvature bounds from below

L. Ambrosio

Scuola Normale Superiore, Pisa
http://cvgmt.sns.it

A.-Gigli-Savaré: *Calculus and heat flow in metric measure spaces* (X, d, m) *and applications to spaces with Ricci bounds from below.*

Goal. To develop a "calculus" in metric measure spaces, use it to identify different notions of heat flow, and apply these results to the Lott-Sturm-Villani metric measure spaces with Ricci curvature bounded from below.

A.-Gigli-Savaré: *Metric measure spaces with Riemannian Ricci curvature bounded from below.*

Goal. Introduce a more restrictive and "Riemannian" notion of Ricci bound from below for metric measure spaces, still consistent and stable under measured Gromov-Hausdorff limits, which rules out Finsler spaces.

(both available on http://cvgmt.sns.it, or ArXiv)

A.-Gigli-Savaré: *Bakry-Emery condition and Riemannian Ricci curvature bounds (to appear).*

Motivations

Cheeger-Colding studied in detail limits, in the Gromov-Hausdorff sense, of sequences of Riemannian manifolds with given dimension N and uniform lower bound K on Ricci tensor (with more recent contributions by Colding-Naber, Honda). Even though many results (rectifiability, tangent spaces, etc.) are available, these limits are described only in metric terms. Question: is there an intrinsic/richer description of these spaces? Can we develop intrinsic calculus tools (gradient, differential, heat flow,..)?

Can we relate the “Lagrangian" $CD(K,N)$ theory, developed by Lott-Sturm-Villani, to the “Eulerian" $cd(\rho,n)$ theory of Bakry-Emery?

One of the great merits of the first theory, based on optimal transportation, is stability under GH limits, while the second one, based on the theory of Markov semigroups and the so-called Γ-calculus, is maybe more powerful in the derivation in sharp form of analytic and geometric inequalities (Poincaré, logarithmic Sobolev, isoperimetric..).

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Outline

1. Some by now "classical" results
2. Identification of weak gradients
3. Identification of gradient flows
4. Riemannian Ricci lower bounds

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Some by now "classical" results

Let us consider in $\mathbb{R}^n$ the heat equation ($u_t(x) = u(t,x)$)

$$\frac{d}{dt}u_t = \Delta u_t.$$

Classically, it can be viewed as the gradient flow of the energy

$$\mathrm{Dir}(u) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx \qquad (+\infty \text{ if } u \notin H^1(\mathbb{R}^n))$$

in the Hilbert space $H = L^2(\mathbb{R}^n)$.

Formally, $t \mapsto u_t$ solves the ODE $u' = -\nabla \mathrm{Dir}(u)$ in H because

$$\mathrm{Dir} \text{ "differentiable" at } u \qquad \Longleftrightarrow \qquad -\Delta u \in L^2,\ \nabla \mathrm{Dir}(u) = -\Delta u.$$

In 1998, Jordan-Kinderlehrer-Otto proved that the same equation arises as gradient flow of the *entropy* functional

$$\mathrm{Ent}(\rho\mathscr{L}^n) := \int_{\mathbb{R}^n} \rho \log \rho \, dx \qquad (+\infty \text{ if } \mu \text{ is not a.c. w.r.t. } \mathscr{L}^n)$$

in the space $\mathscr{P}_2(\mathbb{R}^n)$ of probability measures with finite quadratic moments, with respect to Wasserstein distance W_2.

$$W_2^2(\mu,\nu) := \min\left\{\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2 \, d\gamma(x,y) : \ (\pi_1)_\sharp\gamma = \mu, \ (\pi_2)_\sharp\gamma = \nu\right\}.$$

Push forward notation. $f : X \to Y$ Borel induces a map $f_\sharp : \mathscr{P}(X) \to \mathscr{P}(Y)$:

$$f_\sharp\mu(B) := \mu\big(f^{-1}(B)\big) \qquad \forall B \in \mathscr{B}(Y).$$

The reason underlying the JKO result is that we may view $\mathscr{P}_2(\mathbb{R}^n)$ as an infinite-dimensional differentiable manifold, considering the tangent vector field v_t to a curve (μ_t) in $\mathscr{P}_2(\mathbb{R}^n)$ as the “velocity” occurring in the continuity equation

$$\frac{d}{dt}\mu_t + \nabla\cdot(v_t\mu_t) = 0, \qquad v_t = \nabla\phi_t.$$

Then, defining the metric at $\mu \in \mathscr{P}_2(\mathbb{R}^n)$ as

$$\langle v, w\rangle_\mu := \int v(x)\cdot w(x)\,d\mu(x) \qquad v,\, w \in \overline{\{\nabla\phi:\ \phi\in C^\infty_c(\mathbb{R}^n)\}}^{L^2(\mu)}$$

we turn $\mathscr{P}_2(\mathbb{R}^n)$ into a Riemannian manifold and it can be proved (Otto, Benamou-Brenier) that the induced distance is precisely W_2.
This discovery originated a huge literature, where many other diffusion (even of fourth order) and transport equations are viewed that way, with new existence and uniqueness results, rates of convergence to equilibrium, etc. In our papers we explore the potential of these ideas in a nonsmooth setting.

Proofs of this equivalence

1. By the so-called Otto calculus, i.e. formally viewing $\mathscr{P}(\mathbb{R}^n)$ as an infinite dimensional Riemannian manifold. Computing with this structure the gradient flow of Ent for $\mu_t = \rho_t \mathscr{L}^n$ gives $v_t = \nabla \log \rho_t$.

2. Prove that the implicit time discretization scheme (Euler scheme), traditionally used for the time discrete approximation of gradient flows, when done with energy Ent and distance W_2, does converge to the heat equation.

3. Give a meaning to what "gradient flow of Ent in $\mathscr{P}(\mathbb{R}^n)$ w.r.t. W_2 means", and check that solutions of this gradient flow are solutions to the heat equation. Then, apply uniqueness for $\frac{d}{dt} u_t = \Delta u_t$.

The last strategy is more abstract, but still uses the differentiable structure of $\mathbf{R}^n$. The question is to understand deeper reasons for this equivalence, in particular on which structural properties of the space it depends (Riemannian manifolds, Finsler spaces, Wiener spaces, sub-Riemannian spaces, Alexandrov spaces, etc.)

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Metric measure spaces

Let us consider a *metric measure space* (X, d, m), with $m \in \mathscr{P}(X)$. In this framework it is still possible to define a "Dirichlet energy", that we call Cheeger functional:

$$\mathrm{Ch}(f) := \frac{1}{2} \inf \left\{ \liminf_{n\to\infty} \int_X |\nabla f_n|^2 \, dm : \; f_n \in \mathrm{Lip}(X), \; \int_X |f_n - f|^2 \, dm \to 0 \right\},$$

where

$$|\nabla g|(x) := \limsup_{y\to x} \frac{|g(y) - g(x)|}{d(y,x)}$$

is the *slope* (also called local Lipschitz constant).
Also, one can consider Shannon's *relative entropy functional* $\mathrm{Ent}_m : \mathscr{P}(X) \to [0, +\infty]$

$$\mathrm{Ent}_m(\rho m) := \int_X \rho \log \rho \, dm \qquad (+\infty \text{ if } \mu \text{ is not a.c. w.r.t. } m).$$

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The basic result is that the equivalence between L^2-gradient flow of Ch and W_2-gradient flow of Ent_m *always* holds, if the latter is properly understood. But, without additional assumptions on the space, both objects can be trivial.

Example. Let $X = [0,1]$, d the Euclidean distance, $m = \sum_{n\geq 1} 2^{-n}\delta_{q_n}$, where $\{q_n\}_{n\geq 1}$ is an enumeration of $[0,1]\cap\mathbb{Q}$. Let $A_n \supset \mathbb{Q}\cap X$ be open sets with $\mathscr{L}^1(A_n) \to 0$ and

$$\chi_n(t) := \int_0^t \left(1 - \chi_{A_n}(s)\right) ds \qquad t \in [0,1].$$

Then $f \circ \chi_n \to f$ in $L^2(X,m)$ for all $f \in \mathrm{Lip}(X)$ and $f \circ \chi_n$ is locally constant in $\mathbb{Q}\cap X$ hence

$$\mathrm{Ch}(f) = 0 \qquad \forall f \in \mathrm{Lip}(X).$$

It follows that $\mathrm{Ch} \equiv 0$ in $L^2(X,m)$.

Identification of weak gradients

A closely related question, relevant in particular for the second paper, is the identification of weak gradients. The first one, that we call *relaxed* gradient $|\nabla f|_*$, is the object that provides integral representation to Ch:

$$\mathrm{Ch}(f) = \frac{1}{2}\int_X |\nabla f|_*^2\, dm \qquad \forall f \in D(\mathrm{Ch}).$$

It has all the natural properties a weak gradient should have, for instance locality

$$f = g \text{ on } B \qquad \Longrightarrow \qquad |\nabla f|_* = |\nabla g|_* \ m\text{-a.e. in } B$$

and chain rule

$$|\nabla(\phi \circ f)|_* = |\phi'(f)||\nabla f|_* \qquad m\text{-a.e. in } X.$$

This gradient is useful when doing “vertical” variations $\epsilon \mapsto f + \epsilon g$ (i.e. in the *dependent* variable).

Identification of weak gradients

But, when computing variations of the entropy, the "horizontal" variations $\epsilon \to f(\gamma_\epsilon)$ (i.e. in the *independent* variable) are necessary. These are related to another weak gradient $|\nabla f|_w$, defined as follows.

Let us recall, first, the notion of *upper gradient* (Heinonen-Koskela): it is a function G satisfying

$$(*) \qquad |f(\gamma_1) - f(\gamma_0)| \leq \int_\gamma G$$

on all absolutely continuous curves γ. Obviously $G \geq |\nabla f|$ in a "smooth" setting and the smallest upper gradient is precisely $|\nabla f|$.

We consider the so-called *weak* upper gradient property by requiring (*) along "almost all" curves γ in $AC^2([0,1];X)$. Then, we define $|\nabla f|_w$ as the weak upper gradient G with smallest $L^2(X,m)$ norm. This is related to a notion introduced by Koskela-MacManus, Shanmugalingham, but with a different notion of null set of curves.

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Null sets of curves

We say that a (Borel) set Γ of absolutely continuous curves $\gamma : [0,1] \to X$ is null if

$$\boldsymbol{\pi}(\Gamma) = 0 \qquad \text{for any test plan } \boldsymbol{\pi}.$$

Here, the class of test plans is simply the collection of all $\boldsymbol{\pi} \in \mathscr{P}(AC^2([0,1];X))$ satisfying

$$(e_t)_\sharp \boldsymbol{\pi} \le Cm \ \ \forall t \in [0,1] \qquad \text{for some } C = C(\boldsymbol{\pi}) \ge 0.$$

Theorem. *In any complete and separable metric measure space (X,d,m) with m finite on bounded sets the relaxed gradient $|\nabla f|_*$ and the minimal weak upper gradient $|\nabla f|_w$ coincide m-a.e. in X.*

Of course, maybe they are both trivial without extra assumptions. The proof of this identification uses ideas from optimal transportation, as lifting of solutions to the heat flow to probability measures in $AC^2([0,1];X)$ and the energy dissipation rate of Ent_m.

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Sketch of the proof

First, it is not hard to show that $|\nabla f|_* \geq |\nabla f|_w$, the nontrivial implication is the converse one. To achieve it, we compute the energy dissipation rate of the entropy first in the “Eulerian" way

$$
\begin{aligned}
-\frac{d}{dt}\int_X g_t \log g_t \, dm &= -\int_X \log g_t \Delta g_t \, dm = \int_X \frac{|\nabla g_t|_*^2}{g_t} \, dm \\
&= 4\int_X |\nabla \sqrt{g_t}|_*^2 \, dm.
\end{aligned}
$$

and then we estimate in the “Lagrangian" way

$$
-\frac{d}{dt}\int_X g_t \log g_t \, dm \leq 2\int_X |\nabla \sqrt{g_t}|_*^2 \, dm + 2\int_X |\nabla \sqrt{g_t}|_w^2 \, dm.
$$

Therefore, along solutions to the heat flow, we have the converse inequality $\int |\nabla \sqrt{g_t}|_*^2 \, dm \leq \int |\nabla \sqrt{g_t}|_w^2 \, dm$.

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Equivalence of gradient flows

We assume in this section that the metric measure space (X, d, m) satisfies

$$\int_X e^{-V^2(x)}\, dm(x) < \infty$$

for some Lipschitz weight function $V : X \to \mathbb{R}$. It surely holds with $V(x) = d(x, x_0)$ if $m(B(x_0, r)) \leq Ce^{c\,r^2}$. We also define the *descending slope* of the entropy

$$|\nabla^- \mathrm{Ent}_m|(\mu) := \limsup_{\nu \to \mu} \frac{[\mathrm{Ent}_m(\mu) - \mathrm{Ent}_m(\nu)]^+}{W_2(\nu, \mu)}$$

and we assume that the conditions

$$\sup_n \mathrm{Ent}_m(\rho_n m) < \infty, \quad \rho_n m \rightharpoonup \rho m, \quad \lim_{n\to\infty} \int_X V^2 \rho_n\, dm = \int_X V^2 \rho\, dm,$$

imply

$$\liminf_{n\to\infty} |\nabla^- \mathrm{Ent}_m|(\rho_n m) \geq |\nabla^- \mathrm{Ent}_m|(\rho m)$$

and that $|\nabla^- \mathrm{Ent}_m|$ is an upper gradient of Ent_m. These properties of the slope are fulfilled in all $CD(K, \infty)$ spaces.

Lott-Sturm-Villani $CD(K,\infty)$ spaces

In these spaces (I consider only the case $N = \infty$) one requires K-convexity along Wasserstein geodesics, namely for all $\mu_0,\ \mu_1 \in D(\mathrm{Ent}_m)$ there *exists* a constant speed geodesic μ_t satisfying

$$\mathrm{Ent}_m(\mu_t) \le (1-t)\mathrm{Ent}_m(\mu_0) + t\mathrm{Ent}_m(\mu_1) - \frac{K}{2}t(1-t)W_2^2(\mu_0,\mu_1).$$

When (X,d) is a Riemannian manifold, $CD(K,\infty)$ holds iff $\mathrm{Ric}_X \ge KI$ (Cordero-McCann-Schmuckenschläger, Sturm-Von Renesse).

Consequences of convexity:

• Duality formula for the slope (here stated for $K = 0$):

$$|\nabla^-\mathrm{Ent}_m|(\mu) = \sup_{\nu\neq\mu} \frac{[\mathrm{Ent}_m(\mu) - \mathrm{Ent}_m(\nu)]^+}{W_2(\mu,\nu)}.$$

It implies, among other things, that $\mu \mapsto |\nabla^-\mathrm{Ent}_m|(\mu)$ is l.s.c.

• Upper gradient property. The previous formula for the slope implies a *one-sided and local* Lipschitz estimate

$$\mathrm{Ent}_m(\mu_t) - \mathrm{Ent}_m(\mu_s) \le |\nabla^-\mathrm{Ent}|(\mu_t)W_2(\mu_t,\mu_s).$$

Equivalence of gradient flows

Theorem. *Let $\rho_0 \in L^2(X, m)$ be a probability density and let ρ_t be the L^2-gradient flow of* Ch *starting from ρ_0. Then ρ_t is a probability density for all $t \geq 0$ and $\rho_t m$ is the W_2-gradient flow of Ent_m starting from $f_0 m$.*

Conversely, if $\mu_0 = \rho_0 m$ with $\rho_0 \in L^2(X, m)$ and μ_t is the W_2-gradient flow of Ent_m starting from μ_0, then $\mu_t = \rho_t m$ for all $t \geq 0$.

Finally, the energy dissipation rates coincide:

$$4 \int_X |\nabla \sqrt{\rho_t}|_*^2 \, dm = |\nabla^- \mathrm{Ent}_m|^2(\rho_t m) \qquad \text{for a.e. } t > 0.$$

Corollary. (Fisher information functional and slope of Ent_m coincide)

$$4 \int_X |\nabla \sqrt{\rho}|_*^2 \, dm = |\nabla^- \mathrm{Ent}_m|^2(\rho m).$$

Equivalence of gradient flows

The proof of the theorem consists of the following two steps:

(1) inclusion of L^2-gradient flows into W_2-gradient flows;
(2) uniqueness of W_2-gradient flows.

This strategy, borrowed from Gigli-Kuwada-Ohta, reverses the usual one adopted in $\mathbb{R}^n$, Riemannian manifolds and other "smooth" spaces.

Part (2), a key point in the new strategy, is due to Gigli. Notice however that contractivity of W_2 may fail (Ohta-Sturm) (an open problem is to find whether contractivity holds for other better adapted distances).

I will now focus on the meaning of L^2- and W_2- gradient flows and explain briefly (1).

The L^2-gradient flow

The L^2 gradient flow f_t is the unique solution to the differential inclusion

$$\frac{d}{dt} f_t \in -\partial \mathrm{Ch}(f_t) \qquad \text{for a.e. } t > 0,$$

where

$$\partial \mathrm{Ch}(f) := \left\{ \xi \in L^2(X,m) : \mathrm{Ch}(g) \geq \mathrm{Ch}(f) + \int_X \xi (g-f)\,dm \quad \forall g \in L^2(X,m) \right\}$$

is the subdifferential of the convex and lower semicontinuous functional Ch in $L^2(X,m)$. Existence, uniqueness, L^2-contractivity and the ODE formulation

$$\frac{d}{dt} f_t = \Delta f_t$$

where, by definition, $-\Delta f$ is the element with minimal L^2 norm in $\partial \mathrm{Ch}(f)$, are ensured by the standard Hilbertian theory (Komura, Brezis, Crandall, Pazy). **Warning**: (*) is, in general, a nonlinear PDE!!

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The W_2-gradient flow

The energy dissipation rate for Ent_m along L^2-gradient flows is

$$\frac{d}{dt}\int_X \rho_t \log \rho_t dm = \int_X \log \rho_t \Delta \rho_t \, dm = -\int_{\{\rho_t>0\}} \frac{|\nabla \rho_t|_*^2}{\rho_t}\, dm.$$

The notion of W_2-gradient flow of Ent_m, instead, is based on the idea, due to De Giorgi (and developed in the 80s by Marino, De Giovanni,...), that even for general energies F in general metric spaces all differential informations can be encoded by looking just at the *maximal* rate of energy dissipation:

$$\text{(DG)} \qquad \frac{d}{dt}F(x(t)) \leq -\frac{1}{2}|\nabla F|^2(x(t)) - \frac{1}{2}|x'(t)|^2.$$

Indeed, in a sufficiently smooth setting, along *any* curve $y(t)$, we have

$$\begin{aligned}
&\frac{d}{dt}F(y(t)) = \langle \nabla F(y(t)), y'(t)\rangle \\
\geq\ & -|\nabla F(y(t))||y'(t)| \quad (= \text{iff } -y'(t) \text{ is parallel to } \nabla F(y(t))) \\
\geq\ & -\frac{1}{2}|\nabla F|^2(y(t)) - \frac{1}{2}|y'(t)|^2 \quad (= \text{iff } |\nabla F|(y(t)) = |y'(t)|).
\end{aligned}$$

The W_2-gradient flow

(DG) $$\frac{d}{dt}F(x(t)) \le -\frac{1}{2}|\nabla F|^2(x(t)) - \frac{1}{2}|x'(t)|^2.$$

All terms in (DG) make sense in a metric space (Y, d_Y): $|x'|$ can be replaced by the *metric derivative*

$$|x'|(t) := \lim_{s\to t} \frac{d_Y(x(s), x(t))}{|s-t|}$$

and $|\nabla F|$ by the *descending slope* $|\nabla^- F|$, so that the speed is 0 at minimum points.
Coming back to the case $(Y, d_Y) = (\mathscr{P}_2(X), W_2)$, $F = \mathrm{Ent}_m$, to convert L^2-heat flows to W_2-gradient flows we need to bound both the metric derivative of $t \mapsto \rho_t m$ and the descending slope of $\mathrm{Ent}_m(\rho_t)$ with the L^2-energy dissipation rate.

Kuwada's lemma (from Gigli-Kuwada-Ohta '10)

Lemma. *Let $\rho_0 \in L^2(X, m)$ a probability density, ρ_t the L^2-gradient flow starting from ρ_0. Then the curve $\mu_t := \rho_t m$ is absolutely continuous in $\mathscr{P}_2(X)$ and*

$$|\dot{\mu}_t|^2 \leq \int_{\{\rho_t > 0\}} \frac{|\nabla \rho_t|_*^2}{\rho_t}\, dm \qquad \textit{for a.e. } t > 0.$$

The proof of the Lemma, that we extended to *all* metric measure spaces, requires a fine analysis of the differentiability properties of solutions $Q_t f$ of the Hopf-Lax semigroup

$$Q_t f(x) := \inf_{y \in X} f(y) + \frac{1}{2t} d^2(x, y), \qquad \frac{d}{dt} Q_t f(x) + \frac{1}{2} |\nabla Q_t f|^2(x) \leq 0.$$

These solutions describe (Bernard-Buffoni, Lott-Villani) the evolution in time of Kantorovich potentials.

Riemannian Ricci lower bounds

As shown by Cordero Erausquin-Sturm-Villani, all Minkowski spaces ($\mathbb{R}^n$ endowed with the Lebesgue measure and any norm $\|\cdot\|$) satisfy the $CD(0,n)$ (and therefore the $CD(0,\infty)$) condition. On the other hand, Cheeger-Colding ruled out the possibility to obtain these spaces as limits of Riemannian manifolds.

Question. Is there a more restrictive notion, still stable and (strongly) consistent with the Riemannian case, that rules out Minkowski (and then Finsler) spaces?

Definition. We say that (X,d,m), with $m(X)<\infty$, has *Riemannian Ricci curvature* bounded from below by $K\in\mathbb{R}$, and write $RCD(K,\infty)$, if one of the following equivalent conditions hold:

(i) (X,d,m) is a $CD(K,\infty)$ space and the L^2 heat flow $\mathbf{h}_t$ is linear;

(ii) (X,d,m) is a $CD(K,\infty)$ space and the W_2 heat flow $\mathbf{H}_t$ is additive (i.e. convex and concave) on $\mathscr{P}_2(X)$;

(iii) for all $\mu\in\mathscr{P}_2(X)$ with $\operatorname{supp}\mu\subset\operatorname{supp} m$, $\mathbf{H}_t\mu$ is a gradient flow in the EVI_K sense.

Properties of $RCD(K, \infty)$ spaces

• *Stability under measured Gromov-Hausdorff limits*. Here we can work with the same notions of isomorphim between metric measure spaces and distance between isomorphism classes introduced by Sturm. In the proof of this result it is the EVI_K formulation that plays a decisive role.

• *Tensorization*. If (X, d_X, m_X) and (Y, d_Y, m_Y) are $RCD(K, \infty)$, so is $(X \times Y, \sqrt{d_X^2 + d_Y^2}, m_X \times m_Y)$. Here we can remove the non branching assumption of the $CD(K, N)$ theory.

• *Fine properties of the heat flow*. The identification between the L^2 heat flow $\mathbf{h}_t$ and the W_2 heat flow $\mathbf{H}_t$ allows to pick the best properties from each of them: for instance, the symmetry of the transition probabilities $\theta_t : X \times X \to [0, \infty)$, defined by $\mathbf{H}_t \delta_x := \theta_t(x, \cdot)m$, comes from the fact that $\mathbf{h}_t$ is L^2-selfadjoint, while the contractivity properties of $\mathbf{h}_t$ in spaces different from $L^p(X, m)$ follow from those of $\mathbf{H}_t$.

More properties of the heat flow

(1) The pointwise formula $\tilde{\mathbf{h}}_t f(x) := \int f \, d\mathbf{H}_t \delta_x$ provides a version of $\mathbf{h}_t f$ and an extension of $\mathbf{h}_t$ to a contraction semigroup in all $L^p(X, m)$ spaces.

(2) $\tilde{\mathbf{h}}_t$ leaves $\mathrm{Lip}(\mathrm{supp}\, m)$ invariant and $\mathrm{Lip}(\tilde{\mathbf{h}}_t f) \le e^{-Kt}\mathrm{Lip}(f)$ (it follows by the contractivity estimate $W_2(\mathbf{H}_t \delta x, \mathbf{H}_t \delta_y) \le e^{-Kt} d(x, y)$). Furthermore, $\tilde{\mathbf{h}}_t$ maps $L^\infty(X, m)$ in $C_b(\mathrm{supp}\, m)$.

(3) The Bakry-Emery estimate holds:

$$(\mathrm{BE}_{K,\infty}) \qquad |\nabla(\mathbf{h}_t f)|_*^2 \le e^{-2Kt} \mathbf{h}_t |\nabla f|_*^2 \qquad m\text{-a.e. in } X.$$

Properties of $RCD(K,\infty)$ spaces: Dirichlet forms

Since Ch is a quadratic form in $RCD(K,\infty)$ spaces, the analysis of the connection with Fukushima's theory of Dirichlet forms is mandatory. Let

$$\mathcal{E}(u,v) := \frac{1}{4}\big(\mathrm{Ch}(u+v) - \mathrm{Ch}(u-v)\big)$$

be the bilinear form associated to Ch. It is a Dirichlet form (i.e. closable and Markovian) because Ch is $L^2(X,m)$-lower semicontinuous and decreases, by chain rule, under left composition with 1-Lipschitz maps.

In this theory, two objects are naturally defined, namely the *local energy measure*

$$[u](\varphi) := \mathcal{E}(u,u\varphi) - \mathcal{E}(\frac{u^2}{2},\varphi)$$

and the induced distance

$$d_{\mathcal{E}}(x,y) := \sup\left\{|\psi(x)-\psi(y)| :\ [\psi] \le m\right\}.$$

Properties of $RCD(K, \infty)$ spaces: Dirichlet forms

Theorem. *In a $RCD(K, \infty)$ space (X, d, m) the local energy measure $[u]$ coincides with $|\nabla u|_*^2 m$ and the induced distance $d_{\mathcal{E}}$ coincides with d.*

The proof involves the construction of a symmetric bilinear form

$$(u, v) \in \left[D(\mathrm{Ch})\right]^2 \mapsto \nabla u \cdot \nabla v \in L^1(X, m)$$

satisfying the Leibnitz rule and providing integral representation to $\mathcal{E}$, namely $\mathcal{E}(u, v) = \int \nabla u \cdot \nabla v \, dm$.

In addition, since $\mathcal{E}$ is also strongly local, the theory of Dirichlet forms (Fukushima, Ma-Röckner) can be applied to obtain a unique (in law) *Brownian motion* in $(\mathrm{supp}\, m, d, m)$, i.e. a Markov process $\boldsymbol{X}_t$ with continuous sample paths satisfying

$$\mathbf{P}(\boldsymbol{X}_t | \boldsymbol{X}_0 = x) = \mathbf{H}_t \delta_x \qquad \forall x \in \mathrm{supp}\, m, \; t \geq 0.$$

Equivalence between $(BE)_{K,\infty}$ and $RCD(K,\infty)$

We have seen that $RCD(K,\infty)$ implies the Bakry-Emery condition. We proved recently that also the converse holds.

Theorem. *Let* (X,d,m) *be a metric measure space with* m *satisfying* $\int_X e^{-V^2}\,dm \le 1$, V *Lipschitz. Assume that:*

(i) (X,d) *is complete, separable and length;*

(ii) $\mathrm{supp}\, m = X$ *and Cheeger's energy* Ch *is quadratic;*

(iii) $\mathbf{h}_t$ *maps* $L^\infty(X,m)$ *into* $C_b(X)$ *(Feller);*

(iv) *the (BE) condition holds:*

$$|\nabla \mathbf{h}_t f|_*^2 \le e^{-2Kt}\mathbf{h}_t|\nabla f|_*^2.$$

Then (X,d,m) *is a* $RCD(K,\infty)$ *space.*

The *EVI* formulation of gradient flows

If H is Hilbert and $F : X \to \mathbb{R} \cup \{+\infty\}$ is K-convex and l.s.c., we can write the differential inclusion $-x'(t) \in \partial F(x(t))$ for a.e. $t > 0$ as follows:

$$\forall y,\ \langle -x'(t), y - x(t)\rangle + \frac{K}{2}|(x(t) - y|^2 + F(x(t)) \leq F(y) \quad \text{for a.e. } t > 0.$$

Equivalently

$$\forall y,\ \frac{d}{dt}\frac{1}{2}|x(t) - y|^2 + \frac{K}{2}|(x(t) - y|^2 + F(x(t)) \leq F(y) \quad \text{for a.e. } t > 0.$$

Definition. In a metric space (E, d), a locally absolutely continuous curve $u : (0, \infty) \to E$ is an EVI_K solution to the gradient flow of $F : X \to \mathbb{R} \cup \{+\infty\}$ if for all $v \in D(F)$ it holds

$$\frac{d}{dt}\frac{1}{2}d^2(u(t), v) + \frac{K}{2}|(u(t) - v|^2 F(u(t)) \leq F(v) \quad \text{for a.e. } t > 0.$$

This formulation of gradient flows is equivalent in Hilbert spaces, but in general *stronger* than the one based on energy dissipation (Savaré).

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Open problems and perspectives

(1) For the dimensional theory, i.e. $N < \infty$, we expect similar connections in the case $K = 0$ between convexity of the Renyi entropy

$$\mathcal{E}_N(\rho) := -\int_X \rho^{1-1/N}\, dm \qquad \mu = \rho m + \mu^s,\ \mu_s \perp m$$

the N-dimensional Bakry-Emery condition

$$(\mathrm{BE}_{0,N}) \qquad |\nabla(\mathbf{h}_t f)|_*^2 + \frac{t^2}{N}(\Delta \mathbf{h}_t f)^2 \le \mathbf{h}_t |\nabla f|_*^2 \qquad m\text{-a.e. in } X.$$

and Bochner's inequality

$$\Delta \frac{|\nabla f|_*^2}{2} \ge \langle \nabla \Delta f, \nabla f \rangle + \frac{(\Delta f)^2}{N}.$$

But, the case $CD(K, N)$ with $N < \infty$ and $K \neq 0$ seems to be much more problematic.

(2) What about nonlocal diffusions? Recent work in $\mathbb{R}^n$ by Erbar shows that the Otto equivalence persists, properly understood. In this case W_2 has to be replaced by the distance arising by the minimization of a suitable action functional, in the spirit of Benamou-Brenier.

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(3) In presence of doubling & Poincaré, Cheeger's theory applies and provides, in a suitable and very weak sense, local coordinates and a tangent bundle. The relations with the calculus described in this lecture are still not completely understood.

(4) What about the behaviour on small scales of $RCD(K, \infty)$ spaces? The question makes sense, if one adds a doubling condition on the measure m. The natural conjecture is that tangent metric spaces, in the measured GH sense, are Euclidean. This has been proved by Cheeger-Colding, but for limits of Riemannian manifolds, there is work in progress by Gigli in the $RCD(K, \infty)$ framework.

Thank you for the attention!

SCUOLA NORMALE SUPERIORE