

**Bifurcation from points that are
not in the discrete spectrum of the linearisation**

A degenerate elliptic equation

- Degenerate elliptic b.v.p.
- Finite energy solutions
- Solutions of bvp
- Bvp as an equation in H
- $\langle K(u), v \rangle_A = f'(0) \int_{\Omega} uv dx$, with $f'(0) > 0$
- Bifurcation for bvp

Bifurcation

A degenerate elliptic equation

Degenerate elliptic b.v.p.

$N \geq 3$, $\Omega \subset \mathbb{R}^N$ open bounded, $0 \in \Omega$, $\partial\Omega$ is Lipschitz.

$$\begin{aligned} -\nabla \cdot \{A(x)\nabla u(x)\} &= \lambda f(u(x)) \text{ for } x \in \Omega \\ u(x) &= 0 \text{ for } x \in \partial\Omega, \end{aligned}$$

(D1) $A \in C(\overline{\Omega})$ with $A(x) > 0$ for all $x \in \overline{\Omega} \setminus \{0\}$
and $\lim_{|x| \rightarrow 0} \frac{A(x)}{|x|^2} = \alpha > 0$,

(D2) $f \in C^1(\mathbb{R})$ with $f(0) = 0$, $f'(0) > 0$,
 $\sup\{|f'(s)| : s \in \mathbb{R}\} = M < \infty$.
(f linear is allowed.)

Finite energy solutions

$$E(u) = \frac{1}{2} \int_{\Omega} A |\nabla u|^2 dx - \lambda \int_{\Omega} F(u) dx < \infty$$

where $F(s) = \int_0^s f(t) dt$.

There exist $\alpha_2 \geq \alpha \geq \alpha_1 > 0$ such that $\alpha_1 |x|^2 \leq A(x) \leq \alpha_2 |x|^2$ for all $x \in \bar{\Omega}$.

Hence, $\int_{\Omega} A(x) |\nabla u(x)|^2 dx < \infty \iff \int_{\Omega} |x|^2 |\nabla u|^2 dx < \infty$
and $|F(s)| \leq \frac{M}{2} s^2$,

We seek solutions in the space

$$H = \{u \in L^2 : \int_{\Omega} |x|^2 |\nabla u|^2 dx < \infty, u = 0 \text{ on } \partial\Omega\}.$$

But $\int_{\Omega} u^2 dx \leq \frac{4}{N^2} \int_{\Omega} |x|^2 |\nabla u|^2 dx$ for $u \in H$,
(Hardy for $v(x) = |x| u(x)$).

H is a Hilbert space with equivalent scalar products

$$\langle u, v \rangle_A = \int_{\Omega} A(x) \nabla u \cdot \nabla v dx \text{ and } \langle u, v \rangle_2 = \int_{\Omega} |x|^2 \nabla u \cdot \nabla v dx.$$

H is continuously, but not compactly embedded in $L^2(\Omega)$.

Solutions of bvp

A solution of bvp is a pair $(\lambda, u) \in \mathbb{R} \times H$ such that

$$\int_{\Omega} A(x) \nabla u(x) \cdot \nabla \varphi(x) dx = \lambda \int_{\Omega} f(u(x)) \varphi(x) dx$$

for all $\varphi \in H$

A point $\Lambda \in \mathbb{R}$ is a bifurcation point for bvp
if there is a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times [H \setminus \{0\}]$
of solutions such that

$$\lambda_n \rightarrow \Lambda \text{ and } \|u_n\|_2 \rightarrow 0,$$

$$\iff \|\cdot\|_A \rightarrow 0,$$

since, for a solution

$$\|u_n\|_A = \lambda_n \int_{\Omega} f(u_n) u_n dx \leq |\lambda_n| M \int_{\Omega} u_n^2 dx.$$

$$\Rightarrow E(u_n) \rightarrow 0.$$

Bvp as an equation in H

Define $K(u)$ and $G(u) \in H$ by

$$\langle K(u), v \rangle_A = f'(0) \int_{\Omega} uv dx \text{ and}$$

$$\langle G(u), v \rangle_A = \int_{\Omega} f(u)v dx \text{ for all } u, v \in H$$

$(\lambda, u) \in \mathbb{R} \times H$ satisfies bvp $\Leftrightarrow \langle u, v \rangle_A = \lambda \int_{\Omega} f(u)v dx$ for all $v \in H$
 $\Leftrightarrow u = \lambda G(u)$.

$G : H \rightarrow H$ is H-differentiable at $u = 0$ with $G'(0) = K$.

$G : H \rightarrow H$ is not F-differentiable at 0 if f is not linear.

$G : H \rightarrow H$ is compact if $\lim_{|t| \rightarrow \infty} \frac{f(t)}{t} = 0$. ($\Rightarrow f$ is nonlinear.)

$K \in B(H, H)$ and $K = K^* > 0$.

$\sup \sigma_e(K) = \frac{4f'(0)}{\alpha N^2} > 0$ so $K : H \rightarrow H$ is not compact.

$$\langle K(u), v \rangle_A = f'(0) \int_{\Omega} uv dx, \text{ with } f'(0) > 0$$

$$\sigma(K) \subset [0, \infty) \text{ and } \sup \sigma_e(K) = \frac{4f'(0)}{\alpha N^2}$$

$K = G'(0) : H \rightarrow H$ is not compact.

$$\mu \in \sigma(K) \cap \left(\frac{4f'(0)}{\alpha N^2}, \infty \right) \iff$$

the linear boundary value problem

$$\begin{aligned} -\nabla \cdot \{A(x)\nabla u(x)\} &= \lambda u(x) \text{ for } x \in \Omega \\ u &= 0 \text{ for } x \in \partial\Omega \end{aligned}$$

has a non-trivial solution $u \in H$ for $\lambda = \frac{1}{\mu}$.

$\Sigma = \left\{ \frac{1}{\mu} : \mu \in \sigma(K) \cap \left(\frac{4f'(0)}{\alpha N^2}, \infty \right) \right\}$ consists of eigenvalues of this linearisation of bvp. Σ can be empty.

$$\inf \Sigma = \inf_{u \in H \setminus \{0\}} \frac{\int_{\Omega} A |\nabla u|^2 dx}{\int_{\Omega} u^2 dx}, \quad \inf \Sigma_e = \frac{\alpha N^2}{4f'(0)}.$$

Bifurcation for bvp

Theorem Suppose (D1) and (D2) are satisfied.

Let B be the set of bifurcation points for the bvp.

(i) If $0 \leq f(s)/s \leq f'(0)$ for all $s \neq 0$,
then $B \subset \Sigma \cup [\frac{\alpha N^2}{4f'(0)}, \infty)$.

(ii) If f is odd with $\lim_{|s| \rightarrow \infty} \frac{f(s)}{s} = 0$ and
 $sf(s) < 2 \int_0^s f(t)dt$ for all $s > 0$,
then $\Sigma \cup [\frac{\alpha N^2}{4f'(0)}, \infty) \subset B$.

There is bifurcation to the right at every $\lambda \in \Sigma$,
vertical bifurcation at every $\lambda \in (\frac{\alpha N^2}{4f'(0)}, \infty)$

and $B \cap (0, \infty) = \Sigma \cup [\frac{\alpha N^2}{4f'(0)}, \infty)$.

A degenerate elliptic equation

Bifurcation

- Sufficient conditions for bifurcation without F-diff.
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- Radial solutions
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Bifurcation

Sufficient conditions for bifurcation without F-diff.

$(H, \langle \cdot, \cdot \rangle)$ is a real Hilbert space.

Consider $F = \nabla\psi$, i.e. $\langle F(u), v \rangle = \psi'(u)v$ for all $u, v \in H$

where

(H1) $\psi \in C^1(H, \mathbb{R})$ with $\psi(u) = \psi(-u)$ and $\psi(0) = 0$ such that

$$\lim_{\|u\| \rightarrow \infty} \frac{\psi(u)}{\|u\|^2} = 0$$

and

$$\psi'(u)u < 2\psi(u) \text{ for all } u \in H \setminus \{0\}.$$

(H2) $F : H \rightarrow H$ is compact.

(H3) $F : H \rightarrow H$ is Hadamard differentiable at $u = 0$ with $F'(0) = F'(0)^*$.

Let $\Lambda^e = \sup \sigma_e(F'(0))$.

Bifurcation for $F(u) = \lambda u$

Theorem 3 (with Gilles Evéquoz, 2007) Suppose (H1), (H2) and (H3).

(A) If $\Lambda^e > 0$, then $[0, \Lambda^e] \subset B_F$,

and there is **vertical bifurcation** at every $\lambda \in (0, \Lambda^e)$.

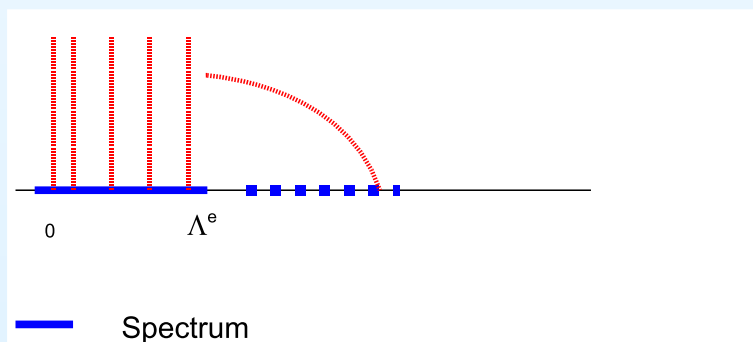
That is, there exists a sequence (λ, u_n) such that $F(u_n) = \lambda u_n$ with $u_n \neq 0$ and $\|u_n\| \rightarrow 0$.

(B) $(\Lambda_+^e, \infty) \cap \sigma(F'(0)) \subset B_F$

where $\Lambda_+^e = \max\{0, \Lambda^e\}$

and there is **bifurcation to the left** at every $\lambda \in (\Lambda_+^e, \infty) \cap \sigma(F'(0))$.

That is, there exists a sequence (λ_n, u_n) such that $F(u_n) = \lambda_n u_n$ with $u_n \neq 0$, $\|u_n\| \rightarrow 0$, $\lambda_n < \lambda$ and $\lambda_n \rightarrow \lambda$.



Remarks

If (H2) holds and F is Fréchet differentiable at $u = 0$, then $F'(0)$ is compact and so $\sigma_e(F'(0)) \subset \{0\}$.

Thus the situation (A) cannot occur in this case.

In (A) we have that $[0, \Lambda^e] \subset B_F$,
without requiring that $(0, \Lambda^e) \subset \sigma(F'(0))$.

We have similar results for equations of the form

$$F(\lambda, u) = 0.$$

The proof is based on Ljusternik-Schnirelman theory.

Idea of the proof of (A)

Fix $\lambda \in (0, \Lambda^e)$ and find critical points of the energy:

$$J(u) = \frac{\lambda}{2} \|u\|^2 - \psi(u). \quad J \text{ is even and } \lim_{\|u\| \rightarrow \infty} J(u) = \infty.$$

Note that $\nabla J(u) = \lambda u - F(u)$.

J satisfies the (P-S) condition since $F : H \rightarrow H$ is compact.

Also $2J(u) - \langle \nabla J(u), u \rangle = \psi'(u)u - 2\psi(u) < 0$ for $u \neq 0$.

Define critical levels using a min-max over sets of genus greater than or equal to n in H : $c_n = \min_{A \in G_n} \max_{u \in A} J(u)$.

F Hadamard diff. at 0 and $\lambda < \sup \sigma_e(F'(0))$

\Rightarrow there exists a small set A_n of genus n such that $\max_{u \in A_n} J(u) < 0$

$\Rightarrow c_n < 0$ for all n .

Also $c_n \rightarrow 0$ as $n \rightarrow \infty$.

(P-S) \Rightarrow there exists u_n such that $J(u_n) = c_n$ and $\nabla J(u_n) = 0$.

$c_n < 0 = J(0) \Rightarrow u_n \neq 0$.

$c_n \rightarrow 0 \Rightarrow \|u_n\| \rightarrow 0$. Hence there is vertical bifurcation at λ .

Another type of nonlinearity

(F) For some $T > 0$, $f \in C^1([-T, T])$ is an odd function that is strictly concave on $[0, T]$ with $f(0) = f(T) = 0$ and $f'(0) > 0$.

Examples: $f(s) = s - |s|^\sigma s$ for any $\sigma > 0$ (sublinear case)
or $f(s) = \sin s$

Set

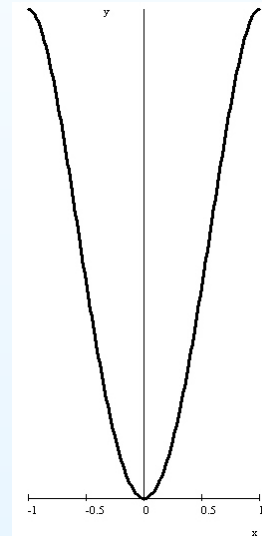
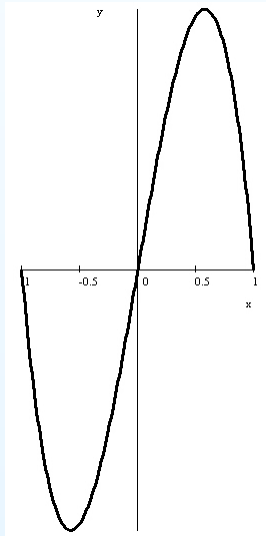
$$F(s) = \int_0^s f(t) dt \text{ for } s \in [-T, T]$$

and extend F to \mathbb{R} as an even function with

$$F \in C^2(\mathbb{R}), F'(s) < 0 \text{ for all } s > T,$$
$$\lim_{s \rightarrow \infty} F(s) = \lim_{s \rightarrow \infty} F'(s) = \lim_{s \rightarrow \infty} F''(s) = 0.$$

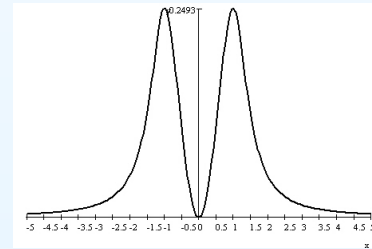
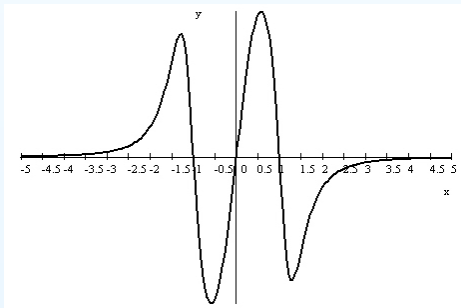
Then $f = F'$ satisfies the condition (D2) of the previous theorems.

Condition (F)



Graphs of f and F

Extension of f



Graphs of the extensions of f and F

Bounded solutions

Theorem Let (D1) and (F) hold. Then $\Sigma \cup \left[\frac{\alpha N^2}{4f'(0)}, \infty \right) \subset B$.

For any $\lambda \in \Sigma \cup \left[\frac{\alpha N^2}{4f'(0)}, \infty \right)$, there exists a sequence of solutions $\{(\lambda_n, u_n)\} \subset (0, \infty) \times [H \setminus \{0\}]$ such that

for all $n \in \mathbb{N}$, $|u_n(x)| \leq T$ a.e. on Ω ,

$\lambda_n \rightarrow \lambda$ and $|u_n|_2 \rightarrow 0$ as $n \rightarrow \infty$.

Since $|u|_1 \leq |\Omega|^{\frac{1}{2}} |u|_2$ it follows that $|u_n|_p \leq |\Omega|^{\frac{1}{2p}} |u_n|_2^{\frac{1}{2}} T^{1-\frac{1}{p}}$.

Hence $|u_n|_p \rightarrow 0$ as $n \rightarrow \infty$ for all $p \in [1, \infty)$.

Radial case

$\Omega = B = \{x \in \mathbb{R}^N : |x| < 1\}$ and $A(x) = C(|x|)$ where
(R) $C \in C^1([0, 1])$ with $C(r) > 0$ for all $r \in (0, 1]$, $C(0) = 0$ and
 $\lim_{r \rightarrow 0} \frac{C'(r)}{r} = 2\alpha$.

Then A satisfies (D1).

Let f satisfy (F).

Then (i) $|u_n|_\infty = T$ for all n

and we have concentration at the origin:

(ii) for any $\varepsilon \in (0, 1)$, $u_n \rightarrow 0$ uniformly on $\{x \in \mathbb{R}^N : \varepsilon \leq |x| \leq 1\}$.

Radial solutions

$$u(x) = v(r) \quad s = r^N \text{ and } w(s) = v(r)$$

BVP becomes

$$-\{D(s)w'(s)\}' = \frac{\lambda}{N^2} f(w(s)) \text{ for } 0 < s < 1$$

where $D(s) = s^{2(1-\frac{1}{N})} C(s^{\frac{1}{N}})$ and

$$w \in X = \{w \in L^2(0, 1) : \int_0^1 s^2 w'(s)^2 ds < \infty \text{ and } w(1) = 0\}$$

Nodal properties

If $\lambda > 0$ and $w \in X \setminus \{0\}$ satisfies

$$-\{D(s)w'(s)\}' = \frac{\lambda}{N^2} f(w(s)) \text{ for } 0 < s < 1$$

then w has only a finite number of zeros in $(0, 1)$

If $\lambda > \frac{\alpha N^2}{4f'(0)}$ and $w \in X$ satisfies

$$-\{D(s)w'(s)\}' = \frac{\lambda}{N^2} f'(0)w(s) \text{ for } 0 < s < 1$$

then w has infinitely many zeros in $(0, 1)$

Weaker degeneracy

(D1)_t $A \in C(\overline{\Omega})$ with $A(x) > 0$ for all $x \in \overline{\Omega} \setminus \{0\}$ and
 $\lim_{|x| \rightarrow 0} \frac{A(x)}{|x|^t} = 1$
for some $t \in [0, 2]$.

We can still define a Hilbert space $(H_A, \langle \cdot, \cdot \rangle_A)$ by

$$H_A = \{u \in L^2(\Omega) : \int_{\Omega} A(x) |\nabla u(x)|^2 dx < \infty \text{ and } u = 0 \text{ on } \partial\Omega\}$$

$$\text{with } \langle u, v \rangle_A = \int_{\Omega} A(x) \nabla u(x) \cdot \nabla v(x) dx.$$

Properties of H_A

Let A satisfy $(D1)_t$ for some $t \in [0, 2]$.

2 Let $H = H_{|\cdot|^2}$.

(i) $(H_A, \langle \cdot, \cdot \rangle_A)$ is continuously embedded in the space $(H_2, (\cdot, \cdot))$ and hence also in $L^2(\Omega)$.

(ii) $(H_A, \langle \cdot, \cdot \rangle_A)$ is continuously embedded in $W^{1,p}(\Omega)$ for $1 \leq p < \frac{2N}{N+t}$.

(iii) $(H_A, \langle \cdot, \cdot \rangle_A)$ is compactly embedded in $L^q(\Omega)$ for $1 \leq q < t^* = \frac{2N}{N+t-2}$.

(iv) If $t = 2$, H_A is NOT compactly embedded in $L^2(\Omega)$.

The boundary-value problem

Let A satisfy $(D1)_t$ for some $t \in [0, 2]$ and $(D2)$.

A solution is a pair $(\lambda, u) \in \mathbb{R} \times H_A$ such that
$$\int_{\Omega} A(x) \nabla u(x) \cdot \nabla \varphi(x) dx = \lambda \int_{\Omega} f(u(x)) \varphi(x) dx$$
 for all $\varphi \in H_A$.

$\Lambda \in \mathbb{R}$ is a bifurcation point if there is a sequence
 $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times [H_A \setminus \{0\}]$ of solutions such that $\lambda_n \rightarrow \Lambda$ and
 $\|u_n\|_2 \rightarrow 0$.

Equation in H_A

As before, define $K(u)$ and $G(u) \in H_A$ by

$$\langle K(u), v \rangle_A = \int_{\Omega} uv dx \text{ and}$$

$$\langle G(u), v \rangle_A = \int_{\Omega} f(u)v dx \text{ for all } u, v \in H_A$$

$(\lambda, u) \in \mathbb{R} \times H_A$ satisfies bvp $\iff u = \lambda G(u)$.

$K \in B(H_A, H_A)$ and $K = K^* > 0$

$G \in C^1(H_A, H_A)$ if $t < 2$

Linearisation

If $t < 2$, $K \in B(H_A, H_A)$ is compact.

$$\sigma(K) = \{\mu_i : i \in \mathbb{N}\} \subset (0, \infty)$$

where $\mu_{i+1} < \mu_i$ and $\lim_{i \rightarrow \infty} \mu_i = 0$

$$\sup \sigma_e(K) = \{0\}$$

The linear boundary value problem

$$-\nabla \cdot \{A(x) \nabla u(x)\} = \lambda u(x) \text{ for } x \in \Omega$$

$$u = 0 \text{ for } x \in \partial\Omega$$

has a non-trivial solution $u \in H_A$ for $\lambda = \frac{1}{\mu_i}$.

$\Sigma = \left\{ \frac{1}{\mu_i} \right\}$ is the set of all eigenvalues.

Regular bifurcation

Theorem Let $(D1)_t$ for some $t \in [0, 2)$ and $(D2)$ be satisfied.

Let B denote the set of all bifurcation points for the bvp.

Then $B = \{\lambda_i : i \in \mathbb{N}\}$

where $0 < \lambda_i < \lambda_{i+1}$ with $\lim_{i \rightarrow \infty} \lambda_i = \infty$.

Recall that for $t = 2$ and under some extra assumptions on f

$$B \cap (0, \infty) = \Sigma \cup \left[\frac{\alpha N^2}{4f'(0)}, \infty \right).$$