UNIVERSITY OF OXFORD
Mathematical Institute

HONOUR SCHOOL OF MATHEMATICS & PHILOSOPHY

SUPPLEMENT TO THE UNDERGRADUATE HANDBOOK – 2013 Matriculation

SYNOPSES OF LECTURE COURSES

Part C
2016-17
For examination in 2017

These synopses can be found at:
https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/handbooks-synopses

Issued October 2016

Handbook for the Undergraduate Mathematics Courses
Supplement to the Handbook
Honour School of Mathematics & Philosophy
Syllabus and Synopses for Part C 2016–2017
for examination in 2017

Contents

1 Foreword

This Supplement to the Mathematics Course Handbook specifies the Mathematics courses available for Part C in Mathematics & Philosophy in the 2017 examination. It should be read in conjunction with the Handbook for Mathematics & Philosophy for the academic year 2016–2017, to be issued in Michaelmas Term. The Handbook contains in particular information on the format and rubrics for written examination papers in Mathematics, and the classification rules applicable to Part C.

See the current edition of the *Examination Regulations* for the full regulations governing the examinations.

1.1 Part C of the Honour School of Mathematics & Philosophy

The following is reproduced from the *Examination Regulations* applicable to the 2017 examinations.

In Part C each candidate shall offer one of the following:

(i) Eight units in Mathematics;

(ii) Six units in Mathematics and one unit in Philosophy;

(iii) Three units in Mathematics and two units in Philosophy;

(iv) Three units in Philosophy;

from the lists for Mathematics and for Philosophy.

The schedule of units in *Mathematics* shall be published in the Mathematics and Philosophy Synopses of lecture courses supplement to the Mathematics Course Handbook by the beginning of the Michaelmas Full Term in the academic year of the examination concerned. No unit in Mathematics, and no subject in Philosophy (apart from the thesis), may be offered in both Part B and Part C.

Further information on the units in Mathematics is given below.

1.2 "Units" and methods of examination

Most courses in Mathematics are assessed by examination. Most subjects offered have a 'weight' of a unit, and will be examined in an examination paper of $1\frac{3}{4}$ hours duration. Each unit paper will contain **3** questions.

1.3 The Schedule of Mathematics units for Mathematics & Philosophy

All units in the Schedule below are drawn from the list of Mathematics Department units and "Other" units available in Mathematics Part C.

In addition you may apply for special approval to be examined in Mathematics Department units not included in the Schedule; any such subject approved will then be treated as falling under the Schedule. For the procedure for seeking approval, see Subsection 1.4 below.

For the 2017 examination, the Schedule is as follows. (N.B. All topics listed are units unless otherwise stated).

Schedule

C1.1 Model Theory

C1.2 Gödel's Incompleteness Theorems

C1.3 Analytic Topology

C1.4 Axiomatic Set Theory

C2.1 Lie Algebras

C2.2 Homological Algebra

C2.3 Representation Theory of Semisimple Lie Algebras

C2.4 Infinite Groups

C2.5 Non-Commutative Rings

C2.6 Introduction to Schemes

C2.7 Category Theory

C3.1 Algebraic Topology

C3.2 Geometric Group Theory

C3.3 Differentiable Manifolds

C3.4 Algebraic Geometry

C3.5 Lie Groups

C3.6 Modular Forms

C3.7 Elliptic Curves

C3.8 Analytic Number Theory

C3.9 Computational Algebraic Topology

C4.1 Functional Analysis

C4.2 Linear Operators

C4.8 Complex Analysis: Conformal Maps and Geometry

C8.1 Stochastic Differential Equations

C8.3 Combinatorics

C8.4 Probabilistic Combinatorics

CCD Dissertations on a Mathematical Topic (double unit)

CCS1 Categories, Proofs and Processes

CCS2 Quantum Computer Science

CCS3 Automata, Logic and Games

CCS4 Advanced Machine Learning [1]

COD Dissertations on a Topic Related to Mathematics (double unit)

And also

Any other unit course from the list of Mathematics Department units for which special approval has been granted.

1.4 Procedure for seeking approval of additional options where this is required

You may, if you have the support of your Mathematics tutor, apply to the Chairman of the Joint Committee for Mathematics and Philosophy for approval of one or more other options from the list of Mathematics Department units for Part C. This list can be found in the Supplement to the Mathematics Course Handbook giving syllabuses and synopses for courses in for Mathematics Part C and at the end of this Supplement.

Applications for special approval must be made through the candidate's college and sent to the Chairman of the Joint Committee for Mathematics and Philosophy, c/o Academic Administrator, Mathematical Institute, to arrive by **Friday of Week 5 of Michaelmas Term**. Be sure to consult your college tutors if you are considering asking for approval to offer one of these additional options.

Given that each of these additional options, which are all in applied mathematics, presume facility with some or other results and techniques covered in first or second year Mathematics courses not taken by Mathematics & Philosophy candidates, such applications will be exceptional. You should also be aware that there may be a clash of lectures for specially approved options and those listed in the above Schedule and with lectures in Philosophy; see the section in The Mathematics Part C Supplement on lecture clashes.

[1] The listed pre-requisites for this course are intended as useful, rather than strict requirements. Students uncertain about the background required for the course should email the lecturer.

1.5 Registration for Part C courses 2016–2017

CLASSES Students will have to register in advance for the courses they wish to take. Students will have to register by Friday of Week 9 of Trinity Term 2016 using the online registration system which can be accessed at https://www.maths.ox.ac.uk/courses/registration/. Students will then be asked to sign up for classes at the start of Michaelmas Term 2016. Further information about this will be sent via email before the start of term.

2 Schedule

2.1 C1.1: Model Theory — Prof. Ehud Hrushovski — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems. A familiarity with (at least the statement of) the Compactness Theorem would also be desirable.

Overview

The course deepens a student's understanding of the notion of a mathematical structure and of the logical formalism that underlies every mathematical theory, taking B1 Logic as a starting point. Various examples emphasise the connection between logical notions and practical mathematics.

The concepts of completeness and categoricity will be studied and some more advanced technical notions, up to elements of modern stability theory, will be introduced.

Learning Outcomes

Students will have developed an in depth knowledge of the notion of an algebraic mathematical structure and of its logical theory, taking B1 Logic as a starting point. They will have an understanding of the concepts of completeness and categoricity and more advanced technical notions.

Synopsis

Structures. The first-order language for structures. The Compactness Theorem for first-order logic. Elementary embeddings. Löwenheim–Skolem theorems. Preservation theorems for substructures. Model Completeness. Quantifier elimination.

Categoricity for first-order theories. Types and saturation. Omitting types. The Ryll Nardzewski theorem characterizing aleph-zero categorical theories. Theories with few types. Ultraproducts.

Reading

1. C.C. Chang and H. Jerome Keisler, *Model Theory* (Third Edition (Dover Books on Mathematics)Paperback)

2. Tent and Ziegler, *A Course in Model Theory*, Cambridge University Press, April 2012

2.2 C1.2: Gödel's Incompleteness Theorems — Dr Dan Isaacson — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

This course presupposes knowledge of first-order predicate logic up to and including soundness and completeness theorems for a formal system of first-order predicate logic (B1 Logic).

Overview

The starting point is Gödel's mathematical sharpening of Hilbert's insight that manipulating symbols and expressions of a formal language has the same formal character as arithmetical operations on natural numbers. This allows the construction for any consistent formal system containing basic arithmetic of a 'diagonal' sentence in the language of that system which is true but not provable in the system. By further study we are able to establish the intrinsic meaning of such a sentence. These techniques lead to a mathematical theory of formal provability which generalizes the earlier results. We end with results that further sharpen understanding of formal provability.

Learning Outcomes

Understanding of arithmetization of formal syntax and its use to establish incompleteness of formal systems; the meaning of undecidable diagonal sentences; a mathematical theory of formal provability; precise limits to formal provability and ways of knowing that an unprovable sentence is true.

Synopsis

Gödel numbering of a formal language; the diagonal lemma. Expressibility in a formal language. The arithmetical undefinability of truth in arithmetic. Formal systems of arithmetic; arithmetical proof predicates. Σ_0-completeness and Σ_1-completeness. The arithmetical hierarchy. ω-consistency and 1-consistency; the first Gödel incompleteness theorem. Separability; the Rosser incompleteness theorem. Adequacy conditions for a provability predicate. The second Gödel incompleteness theorem; Löb's theorem. Provable Σ_1-completeness. The ω-rule. The provability logics GL; fixed point theorems for GL. The Bernays arithmetized completeness theorem; undecidable Δ_2-sentences of arithmetic.

Reading

1. Lecture notes for the course.

Further Reading

1. Raymond M. Smullyan, *Gödel's Incompleteness Theorems* (Oxford University Press, 1992).
2. George S. Boolos and Richard C. Jeffrey, *Computability and Logic* (3rd edition, Cambridge University Press, 1989), Chs 15, 16, 27 (pp 170–190, 268-284).
3. George Boolos, *The Logic of Provability* (Cambridge University Press, 1993).

2.3 C1.3: Analytic Topology — Dr Rolf Suabedissen — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part A Topology; a basic knowledge of Set Theory, including cardinal arithmetic, ordinals and the Axiom of Choice, will also be useful.

Overview

The aim of the course is to present a range of major theory and theorems, both important and elegant in themselves and with important applications within topology and to mathematics as a whole. Central to the course is the general theory of compactness, compactifications and Tychonoff's theorem, one of the most important in all mathematics (with applications across mathematics and in mathematical logic) and computer science.

Synopsis

Bases and initial topologies (including pointwise convergence and the Tychonoff product topology). Separation axioms, continuous functions, Urysohn's lemma. Separable, Lindelöf and second countable spaces. Urysohn's metrization theorem. Filters and ultrafilters. Tychonoff's theorem. Compactifications, in particular the Alexandroff One-Point Compactification and the Stone–Čech Compactification. Completeness, connectedness and local connectedness. Components and quasi-components. Totally disconnected compact spaces. Paracompactness; Bing Metrization Theorem.

Reading

1. S. Willard, *General Topology* (Addison–Wesley, 1970), Chs. 1–8.
2. R. Engelking, *General Topology* (Sigma Series in Pure Mathematics, Vol 6, 1989)

2.4 C1.4: Axiomatic Set Theory — Dr Rolf Suabedissen — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems, together with a course on basic set theory, including cardinals and ordinals, the Axiom of Choice and the Well Ordering Principle.

Overview

Inner models and consistency proofs lie at the heart of modern Set Theory, historically as well as in terms of importance. In this course we shall introduce the first and most important of inner models, Gödel's constructible universe, and use it to derive some fundamental consistency results.

Synopsis

A review of the axioms of ZF set theory. Absoluteness, the recursion theorem. The Cumulative Hierarchy of sets and the consistency of the Axiom of Foundation as an example of the method of inner models. Levy's Reflection Principle. Gödel's inner model of constructible sets and the consistency of the Axiom of Constructibility ($V = L$). $V = L$ is absolute. The fact that $V = L$ implies the Axiom of Choice. Some advanced cardinal arithmetic. The fact that $V = L$ implies the Generalized Continuum Hypothesis.

Reading

For the review of ZF set theory and the prerequisites from Logic:

1. D. Goldrei, *Classic Set Theory* (Chapman and Hall, 1996).
2. K. Kunen, *The Foundations of Mathematics* (College Publications, 2009).

For course topics (and much more):

1. K. Kunen, *Set Theory* (College Publications, 2011) Chapters (I and II).

Further Reading

1. K. Hrbacek and T. Jech, *Introduction to Set Theory* (3rd edition, M Dekker, 1999).

2.5 C2.1: Lie Algebras — Prof. Dan Ciubotaru — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part B course B2.1 Introduction to Representation Theory. A thorough knowledge of linear algebra and the second year algebra courses; in particular familiarity with group actions, quotient rings and vector spaces, isomorphism theorems and inner product spaces will be assumed. Some familiarity with the Jordan–Hölder theorem and the general ideas of representation theory will be an advantage.

Overview

Lie Algebras are mathematical objects which, besides being of interest in their own right, elucidate problems in several areas in mathematics. The classification of the finite-dimensional complex Lie algebras is a beautiful piece of applied linear algebra. The aims of this course are to introduce Lie algebras, develop some of the techniques for studying them, and describe parts of the classification mentioned above, especially the parts concerning root systems and Dynkin diagrams.

Learning Outcomes

Students will learn how to utilise various techniques for working with Lie algebras, and they will gain an understanding of parts of a major classification result.

Synopsis

Definition of Lie algebras, small-dimensional examples, some classical groups and their Lie algebras (treated informally). Ideals, subalgebras, homomorphisms, modules.

Nilpotent algebras, Engel's theorem; soluble algebras, Lie's theorem. Semisimple algebras and Killing form, Cartan's criteria for solubility and semisimplicity, Weyl's theorem on complete reducibility of representations of semisimple Lie algebras.

The root space decomposition of a Lie algebra; root systems, Cartan matrices and Dynkin diagrams. Discussion of classification of irreducible root systems and semisimple Lie algebras.

Reading

1. J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory,* Graduate Texts in Mathematics 9 (Springer-Verlag, 1972, reprinted 1997). Chapters 1–3 are relevant and part of the course will follow Chapter 3 closely.

2. B. Hall, *Lie Groups, Lie Algebras, and Representations. An Elementary Introduction,* Graduate Texts in Mathematics 222 (Springer-Verlag, 2003).

3. K. Erdmann, M. J. Wildon, *Introduction to Lie Algebras* (Springer-Verlag, 2006), ISBN: 1846280400.

Further Reading

1. J.-P. Serre, *Complex Semisimple Lie Algebras* (Springer, 1987). Rather condensed, assumes the basic results. Very elegant proofs.

2. N. Bourbaki, *Lie Algebras and Lie Groups* (Masson, 1982). Chapters 1 and 4–6 are relevant; this text fills in some of the gaps in Serre's text.

3. William Fulton, Joe Harris, *Representation theory: a first course*, GTM, Springer.

2.6 C2.2 Homological Algebra — Prof. Andre Henriques — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part A Rings and Modules. Introduction to Representation Theory B2.1 is recommended but not essential.

Overview

Homological algebra is one of the most important tools in mathematics with application ranging from number theory and geometry to quantum physics. This course will introduce the basic concepts and tools of homological algebra with examples in module theory and group theory.

Learning Outcomes

Students will learn about abelian categories and derived functors and will be able to apply these notions in different contexts. They will learn to compute Tor, Ext, and group cohomology and homology.

Synopsis

Chain complexes: complexes of R-modules, operations on chain complexes, long exact sequences, chain homotopies, mapping cones and cylinders (4 hours) Derived functors: delta functors, projective and injective resolutions, left and right derived functors (5 hours) Tor and Ext: Tor and flatness, Ext and extensions, universal coefficients theorems, Koszul resolutions (4 hours) Group homology and cohomology: definition, interpretation of H^1 and H^2, universal central extensions, the Bar resolution (3 hours).

Reading

Weibel, Charles *An introduction to Homological algebra* (see Google Books)

2.7 C2.3: Representation Theory of Semisimple Lie Algebras — Prof. Dan Ciubotaru —16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Past attendance at Lie algebras is recommended, but not required. Past attendance at Introduction to Representation Theory is recommended as well, but not required.

Overview

The representation theory of semisimple Lie algebras plays a central role in modern mathematics with motivation coming from many areas of mathematics and physics, for example, the Langlands program. The methods involved in the theory are diverse and include remarkable interactions with algebraic geometry, as in the proofs of the Kazhdan-Lusztig and Jantzen conjectures.

The course will cover the basics of finite dimensional representations of semisimple Lie algebras (e.g., the Cartan-Weyl highest weight classification) in the framework of the larger Bernstein-Gelfand-Gelfand category $\mathcal{O}$.

Learning Outcomes

The students will have developed a comprehensive understanding of the basic concepts and modern methods in the representation theory of semisimple Lie algebras, including the classification of finite dimensional modules, the classification of objects in category O, character formulas, Lie algebra cohomology and resolutions of finite dimensional modules.

Synopsis

Universal enveloping algebra of a Lie algebra, Poincaré-Birkhoff-Witt theorem, basic definitions and properties of representations of Lie algebras, tensor products.

The example of $sl(2)$: finite dimensional modules, highest weights.

Category $\mathcal{O}$: Verma modules, highest weight modules, infinitesimal characters and Harish-Chandra's isomorphism, formal characters, contravariant (Shapovalov) forms.

Finite dimensional modules of a semisimple Lie algebra: the Cartan-Weyl classification, Weyl character formula, dimension formula, Kostant's multiplicity formula, examples.

Homological algebra: Lie algebra cohomology, Bernstein-Gelfand-Gelfand resolution of finite dimensional modules, Ext groups in category $\mathcal{O}$.

Topics: applications, Bott's dimension formula for Lie algebra cohomology groups, characters of the symmetric group (via Zelevinsky's application of the BGG resolution to Schur-Weyl duality).

Reading

1. Course Lecture Notes.
2. J. Bernstein, "Lectures on Lie algebras", in *Representation Theory, Complex Analysis, and Integral Geometry* (Springer 2012).

Further reading

1. J. Humphreys, *Representations of semisimple Lie algebras in the BGG category* $\mathcal{O}$ (AMS, 2008).
2. J. Humphreys, *Introduction to Lie algebras and representation theory* (Springer, 1997).
3. W. Fulton, J. Harris, *Representation Theory* (Springer 1991).

2.8 C2.4: Infinite Groups — Prof. Dan Segal — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

A thorough knowledge of the second-year algebra courses; in particular, familiarity with group actions, quotient rings and quotient groups, and isomorphism theorems will be assumed. Familiarity with the Commutative Algebra course will be helpful but not essential.

Overview

The concept of a group is so general that anything which is true of all groups tends to be rather trivial. In contrast, groups that arise in some specific context often have a rich and beautiful theory. The course introduces some natural families of groups, various questions that one can ask about them, and various methods used to answer these questions; these involve among other things rings and trees.

Synopsis

Free groups and their subgroups; finitely generated groups: counting finite-index subgroups; finite presentations and decision problems; Linear groups: residual finiteness; structure of soluble linear groups; Nilpotency and solubility: lower central series and derived series; structural and residual properties of finitely generated nilpotent groups and polycyclic groups; characterization of polycyclic groups as soluble Z-linear groups; Torsion groups and the General Burnside Problem.

Reading

1. D. J. S. Robinson, *A course in the theory of groups*, 2nd ed., Graduate texts in Mathematics, (Springer-Verlag, 1995). Chapters 2, 5, 6, 15.

2. D. Segal, *Polycyclic groups*, (CUP, 2005) Chapters 1 and 2.

2.9 C2.5: Non-Commutative Rings — Prof. Konstantin Ardakov — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Prerequisites: Part A Rings and Modules.
Recommended background: Introduction to Representation Theory B2.1, Part B Commutative Algebra (from 2016 onwards).

Overview

This course builds on Algebra 2 from the second year. We will look at several classes of non-commutative rings and try to explain the idea that they should be thought of as functions on "non-commutative spaces". Along the way, we will prove several beautiful structure theorems for Noetherian rings and their modules.

Learning Outcomes

Students will be able to appreciate powerful structure theorems, and be familiar with examples of non-commutative rings arising from various parts of mathematics.

Synopsis

1. Examples of non-commutative Noetherian rings: enveloping algebras, rings of differential operators, group rings of polycyclic groups. Filtered and graded rings. (3 hours)

2. Jacobson radical in general rings. Jacobson's density theorem. Artin-Wedderburn. (3 hours)

3. Ore localisation. Goldie's Theorem on Noetherian domains. (3 hours)

4. Minimal prime ideals and dimension functions. Rees rings and good filtrations. (3 hours)

5. Bernstein's Inequality and Gabber's Theorem on the integrability of the characteristic variety. (4 hours)

Reading

1. K.R. Goodearl and R.B. Warfield, *An Introduction to Noncommutative Noetherian Rings* (CUP, 2004).

Further reading

1. M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra* (Westview Press, 1994).
2. S.C. Coutinho, *A Primer of Algebraic D-modules* (CUP, 1995).
3. J. Björk, *Analytic D-Modules and Applications* (Springer, 1993).

2.10 C2.6: Introduction to Schemes — Prof. Damian Rossler — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Commutative Algebra is essential. Homological Algebra is highly recommended and Category Theory is recommended but the necessary material from both courses can be learnt during the course (see the beginning of the lecture notes for precise references). Algebraic Geometry is recommended but not technically necessary. Algebraic Topology contains many homological techniques also used in C2.6.

Overview

Scheme theory is the foundation of modern algebraic geometry. It unifies algebraic geometry with algebraic number theory. This unification has led to proofs of important conjectures in number theory such as the Weil conjecture by Deligne and the Mordell conjecture by Faltings.

This course will cover the basics of the theory of schemes, with an emphasis on cohomological techniques.

Learning Outcomes

Students will have developed a thorough understanding of the basic concepts and methods of scheme theory. They will be able to work with affine and projective schemes, as well as with coherent sheaves and their cohomology groups.

Synopsis

Sheaves and cohomology of sheaves.

Affine schemes: points, topology, structure sheaf. Schemes: definition, subschemes, morphisms, glueing. Relative schemes: fibred products, Cohomological characterisation of affine schemes.

Projective schemes, morphisms to projective space. Ample line bundles. Cohomological characterisation of ampleness.

Flat morphisms, semicontinuity, Hilbert polynomials. Cohomological characterisation of flatness.

Constructibility and irreducibility. Images of constructible sets.

Separatedness, properness and valuative criteria. Hilbert and Quot schemes.

Reading

1. Robin Hartshorne, *Algebraic Geometry.*
2. Ravi Vakil, *Foundations of Algebraic Geometry*, online notes on the website of Stanford University (open access).

Further reading

1. David Mumford, *The Red Book of Varieties and Schemes.*
2. David Eisenbud and Joe Harris, *The Geometry of Schemes.*

2.11 C2.7: Category Theory — Prof. Kobi Kremnitzer — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

There are no essential prerequisites but past attendance at several Part A pure maths options would be very useful, and the Part B courses Topology and Groups and Introduction to Representation Theory are relevant, as are a number of Part C courses. Category Theory also has links with B1 Logic and Set Theory, but this course will not stress those links.

Overview

Category theory brings together many areas of pure mathematics (and also has close links to logic and to computer science). It is based on the observation that many mathematical topics can be unified and simplified by using descriptions in terms of diagrams of arrows; the arrows represent functions of suitable types. Moreover many constructions in pure mathematics can be described in terms of 'universal properties' of such diagrams.

The aim of this course is to provide an introduction to category theory using a host of familiar examples, to explain how these examples fit into a categorical framework and to use categorical ideas to make new constructions.

Learning Outcomes

Students will have developed a thorough understanding of the basic concepts and methods of category theory. They will be able to work with commutative diagrams, naturality and universality properties, and to apply categorical ideas and methods in a wide range of areas of mathematics.

Synopsis

Introduction: universal properties in linear and multilinear algebra.

Categories, functors, natural transformations. Examples including categories of sets, groups, rings, vector spaces and modules, topological spaces. Groups, monoids and partially ordered sets as categories. Opposite categories and the principle of duality. Covariant, contravariant, faithful and full functors.

Adjoints: definition and examples including free and forgetful functors and abelianisations of groups. Adjunctions via units and counits, adjunctions via initial objects.

Representables: definitions and examples including tensor products. The Yoneda lemma and applications.

Limits and colimits, including products, equalizers, pullbacks and pushouts. Monics and epics. Interaction between functors and limits.

Monads and comonads, Barr-Beck monadicity theorem, faithfully flat descent, pure monomorphisms of rings. The category of affine schemes as the opposite of the category of commutative rings. Examples of non-affine schemes.

Reading

1. T. Leinster, *Basic category theory*, (CUP, 2014) Chapters 1-5

Further reading

1. D. Eisenbud, J. Harris, *The geometry of schemes.*

2. S. Lang, *Linear algebra* 2nd edition, (Addison Wesley, 1971) Chapter XIII, out of print but may be available in college libraries.
3. S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., (Springer, 1998)
4. S. Awodey, *Category theory*, Oxford Logic Guides (OUP, 2010)
5. D.G. Northcott, *Multilinear algebra* (CUP, reissued 2009)

2.12 C3.1: Algebraic Topology — Prof. Christopher Douglas — 16MT

Level: M-level. **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Helpful but not essential: Part A Topology, B3.5 Topology and Groups.

Overview

Homology theory is a subject that pervades much of modern mathematics. Its basic ideas are used in nearly every branch, pure and applied. In this course, the homology groups of topological spaces are studied. These powerful invariants have many attractive applications. For example we will prove that the dimension of a vector space is a topological invariant and the fact that 'a hairy ball cannot be combed'.

Learning Outcomes

At the end of the course, students are expected to understand the basic algebraic and geometric ideas that underpin homology and cohomology theory. These include the cup product and Poincaré Duality for manifolds. They should be able to choose between the different homology theories and to use calculational tools such as the Mayer-Vietoris sequence to compute the homology and cohomology of simple examples, including projective spaces, surfaces, certain simplicial spaces and cell complexes. At the end of the course, students should also have developed a sense of how the ideas of homology and cohomology may be applied to problems from other branches of mathematics.

Synopsis

Chain complexes of free Abelian groups and their homology. Short exact sequences. Delta complexes and their homology. Euler characteristic.

Singular homology of topological spaces. Relative homology and the Five Lemma. Homotopy invariance and excision (details of proofs not examinable). Mayer-Vietoris Sequence. Equivalence of simplicial and singular homology.

Degree of a self-map of a sphere. Cell complexes and cellular homology. Application: the hairy ball theorem.

Cohomology of spaces and the Universal Coefficient Theorem (proof not examinable). Cup products. Künneth Theorem (without proof). Topological manifolds and orientability. The fundamental class of an orientable, closed manifold and the degree of a map between manifolds of the same dimension. Poincaré Duality (without proof).

Reading

1. A. Hatcher, *Algebraic Topology* (Cambridge University Press, 2001). Chapters 2 and 3.
2. G. Bredon, *Topology and Geometry* (Springer, 1997). Chapters 4 and 5.
3. J. Vick, *Homology Theory,* Graduate Texts in Mathematics 145 (Springer, 1973).

2.13 C3.2: Geometric Group Theory — Prof. Panos Papazoglou — 16HT

Level: M-level. **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites.

The Topology & Groups course is a helpful, though not essential prerequisite.

Overview.

The aim of this course is to introduce the fundamental methods and problems of geometric group theory and discuss their relationship to topology and geometry.

The first part of the course begins with an introduction to presentations and the list of problems of M. Dehn. It continues with the theory of group actions on trees and the structural study of fundamental groups of graphs of groups.

The second part of the course focuses on modern geometric techniques and it provides an introduction to the theory of Gromov hyperbolic groups.

Synopsis.

Free groups. Group presentations. Dehn's problems. Residually finite groups.

Group actions on trees. Amalgams, HNN-extensions, graphs of groups, subgroup theorems for groups acting on trees.

Quasi-isometries. Hyperbolic groups. Solution of the word and conjugacy problem for hyperbolic groups.

If time allows: Small Cancellation Groups, Stallings Theorem, Boundaries.

Reading.

1. J.P. Serre, *Trees* (Springer Verlag 1978).
2. M. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature, Part III* (Springer, 1999), Chapters I.8, III.H.1, III. *Gamma* 5.
3. H. Short *et al.*, 'Notes on word hyperbolic groups', *Group Theory from a Geometrical Viewpoint, Proc. ICTP Trieste* (eds E. Ghys, A. Haefliger, A. Verjovsky, World Scientific 1990)

   available online at: http://www.cmi.univ-mrs.fr/ hamish/
4. C.F. Miller, *Combinatorial Group Theory*, notes: http://www.ms.unimelb.edu.au/ cfm/notes/cgt-notes.pdf.

Further Reading.

1. G. Baumslag, *Topics in Combinatorial Group Theory* (Birkhauser, 1993).
2. O. Bogopolski, *Introduction to Group Theory* (EMS Textbooks in Mathematics, 2008).
3. R. Lyndon, P. Schupp, *Combinatorial Group Theory* (Springer, 2001).
4. W. Magnus, A. Karass, D. Solitar,*Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations* (Dover Publications, 2004).
5. P. de la Harpe, *Topics in Geometric Group Theory*, (University of Chicago Press, 2000).

2.14 C3.3: Differentiable Manifolds — Prof. Dominic Joyce — 16MT

Level: M-level. **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part A Introduction to Manifolds. Useful but not essential: Part B Geometry of Surfaces.

Overview

A manifold is a space such that small pieces of it look like small pieces of Euclidean space. Thus a smooth surface, the topic of the B3 course, is an example of a (2-dimensional) manifold.

Manifolds are the natural setting for parts of classical applied mathematics such as mechanics, as well as general relativity. They are also central to areas of pure mathematics such as topology and certain aspects of analysis.

In this course we introduce the tools needed to do analysis on manifolds. We prove a very general form of Stokes' Theorem which includes as special cases the classical theorems of Gauss, Green and Stokes. We also introduce the theory of de Rham cohomology, which is central to many arguments in topology.

Learning Outcomes

The candidate will be able to manipulate with ease the basic operations on tangent vectors, differential forms and tensors both in a local coordinate description and a global coordinate-free one; have a knowledge of the basic theorems of de Rham cohomology and some simple examples of their use; know what a Riemannian manifold is and what geodesics are.

Synopsis

Smooth manifolds and smooth maps. Tangent vectors, the tangent bundle, induced maps. Vector fields and flows, the Lie bracket and Lie derivative.

Exterior algebra, differential forms, exterior derivative, Cartan formula in terms of Lie derivative. Orientability. Partitions of unity, integration on oriented manifolds.

Stokes' theorem. De Rham cohomology. Applications of de Rham theory including degree.

Riemannian metrics. Isometries. Geodesics.

Reading

1. M. Spivak, *Calculus on Manifolds*, (W. A. Benjamin, 1965).
2. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 1, (1970).
3. W. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, 2nd edition, (Academic Press, 1986).
4. M. Berger and B. Gostiaux, *Differential Geometry: Manifolds, Curves and Surfaces.* Translated from the French by S. Levy, (Springer Graduate Texts in Mathematics, 115, Springer–Verlag (1988)) Chapters 0–3, 5–7.
5. F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, (Springer Graduate Texts in Mathematics, 1994).
6. D. Barden and C. Thomas, *An Introduction to Differential Manifolds.* (Imperial College Press, London, 2003.)

2.15 C3.4: Algebraic Geometry — Prof. Alexander Ritter — 16MT

Level: M-level. **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part A Rings and Modules. B3.3 Algebraic Curves useful but not essential.

Overview

Algebraic geometry is the study of algebraic varieties: an algebraic variety is roughly speaking, a locus defined by polynomial equations. One of the advantages of algebraic geometry is that it is purely algebraically defined and applied to any field, including fields of finite characteristic. It is geometry based on algebra rather than calculus, but over the real or complex numbers it provides a rich source of examples and inspiration to other areas of geometry.

Synopsis

Affine algebraic varieties, the Zariski topology, morphisms of affine varieties. Irreducible varieties.

Projective space. Projective varieties, affine cones over projective varieties. The Zariski topology on projective varieties. The projective closure of affine variety. Morphisms of projective varieties. Projective equivalence.

Veronese morphism: definition, examples. Veronese morphisms are isomorphisms onto their image; statement, and proof in simple cases. Subvarieties of Veronese varieties. Segre maps and products of varieties. Categorical products: the image of the Segre map gives the categorical product.

Coordinate rings. Hilbert's Nullstellensatz. Correspondence between affine varieties (and morphisms between them) and finitely generated reduced k-algebras (and morphisms between them). Graded rings and homogeneous ideals. Homogeneous coordinate rings.

Categorical quotients of affine varieties by certain group actions. The maximal spectrum.

Primary decomposition of ideals.

Discrete invariants of projective varieties: degree, dimension, Hilbert function. Statement of theorem defining Hilbert polynomial.

Quasi-projective varieties, and morphisms between them. The Zariski topology has a basis of affine open subsets. Rings of regular functions on open subsets and points of quasi-projective varieties. The ring of regular functions on an affine variety is the coordinate ring. Localisation and relationship with rings of regular functions.

Tangent space and smooth points. The singular locus is a closed subvariety. Algebraic re-formulation of the tangent space. Differentiable maps between tangent spaces.

Function fields of irreducible quasi-projective varieties. Rational maps between irreducible varieties, and composition of rational maps. Birational equivalence. Correspondence between dominant rational maps and homomorphisms of function fields. Blow-ups: of affine space at a point, of subvarieties of affine space, and of general quasi-projective varieties along general subvarieties. Statement of Hironaka's Desingularisation Theorem. Every irreducible variety is birational to a hypersurface. Re-formulation of dimension. Smooth points are a dense open subset.

Reading

KE Smith et al, *An Invitation to Algebraic Geometry*, (Springer 2000), Chapters 1–8.

Further Reading

1. M Reid, *Undergraduate Algebraic Geometry*, LMS Student Texts 12, (Cambridge 1988).
2. K Hulek, *Elementary Algebraic Geometry*, Student Mathematical Library 20. (American Mathematical Society, 2003).
3. A Gathmann, *Algebraic Geometry lecture notes*, online: `www.mathematik.uni-kl.de/en/agag/members/professors/gathmann/notes/alggeom`
4. I Shafarevich, *Basic Algebraic Geometry 1*, (Springer, 1994).
5. D Mumford, *The Red Book of Varieties and Schemes*, (Springer, 2009).

2.16 C3.5: Lie Groups — Prof. Andrew Dancer — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part A Group Theory, Topology and Introduction to Manifolds are all useful but not essential.

Overview

The theory of Lie Groups is one of the most beautiful developments of pure mathematics in the twentieth century, with many applications to geometry, theoretical physics and mechanics. The subject is an interplay between geometry, analysis and algebra. Lie groups are groups which are simultaneously manifolds, that is geometric objects where the notion of differentiability makes sense, and the group multiplication and inversion are differentiable maps. The majority of examples of Lie groups are the familiar groups of matrices. The course does not require knowledge of differential geometry: the basic tools needed will be covered within the course.

Learning Outcomes

Students will have learnt the fundamental relationship between a Lie group and its Lie algebra, and the basics of representation theory for compact Lie groups. This will include a firm understanding of maximal tori and the Weyl group, and their role for representations.

Synopsis

Brief introduction to manifolds. Classical Lie groups. Left-invariant vector fields, Lie algebra of a Lie group. One-parameter subgroups, exponential map. Homomorphisms of Lie groups and Lie algebras. Ad and ad. Compact connected abelian Lie groups are tori. The Campbell-Baker-Hausdorff series (statement only).

Lie subgroups. Definition of embedded submanifolds. A subgroup is an embedded Lie subgroup if and only if it is closed. Continuous homomorphisms of Lie groups are smooth. Correspondence between Lie subalgebras and Lie subgroups (proved assuming the Frobenius theorem). Correspondence between Lie group homomorphisms and Lie algebra homomorphisms. Ado's theorem (statement only), Lie's third theorem.

Basics of representation theory: sums and tensor products of representations, irreducibility, Schur's lemma. Compact Lie groups: left-invariant integration, complete reducibility. Representations of the circle and of tori. Characters, orthogonality relations. Peter-Weyl theorem (statement only).

Maximal tori. Roots. Conjugates of a maximal torus cover a compact connected Lie group (proved assuming the Lefschetz fixed point theorem). Weyl group. Reflections. Weyl group of $U(n)$. Representations of a compact connected Lie group are the Weyl-invariant representations of a maximal torus (proof of inclusion only). Representation ring of T^n and $U(n)$.

Killing form. Remarks about the classification of compact Lie groups.

Reading

1. J. F. Adams, *Lectures on Lie Groups* (University of Chicago Press, 1982).
2. T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups* (Graduate Texts in Mathematics, Springer, 1985).

Further Reading

1. R. Carter, G. Segal and I. MacDonald, *Lectures on Lie Groups and Lie Algebras* (LMS Student Texts, Cambridge, 1995).
2. W. Fulton, J. Harris, *Representation Theory: A First Course* (Graduate Texts in Mathematics, Springer, 1991).
3. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Graduate Texts in Mathematics, 1983).

2.17 C3.6: Modular Forms — Prof. Alan Lauder — 16HT

Level: M-Level. **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part A Number Theory, Topology and Part B Geometry of Surfaces, Algebraic Curves are useful but not essential.

Overview

The course aims to introduce students to the beautiful theory of modular forms, one of the cornerstones of modern number theory. This theory is a rich and challenging blend of methods from complex analysis and linear algebra, and an explicit application of group actions.

Learning Outcomes

The student will learn about modular curves and spaces of modular forms, and understand in special cases how to compute their genus and dimension, respectively. They will see that modular forms can be described explicitly via their q-expansions, and they will be familiar with explicit examples of modular forms. They will learn about the rich algebraic structure on spaces of modular forms, given by Hecke operators and the Petersson inner product.

Synopsis

1. Overview and examples of modular forms. Definition and basic properties of modular forms.
2. Topology of modular curves: a fundamental domain for the full modular group; fundamental domains for subgroups Γ of finite index in the modular group; the compact surfaces X_Γ; explicit triangulations of X_Γ and the computation of the genus using the Euler characteristic formula; the congruence subgroups $\Gamma(N), \Gamma_1(N)$ and $\Gamma_0(N)$; examples of genus computations.
3. Dimensions of spaces of modular forms: general dimension formula (proof non-examinable); the valence formula (proof non-examinable).
4. Examples of modular forms: Eisenstein series in level 1; Ramanujan's Δ function; some arithmetic applications.
5. The Petersson inner product.
6. Modular forms as functions on lattices: modular forms of level 1 as functions on lattices; Eisenstein series revisited.
7. Hecke operators in level 1: Hecke operators on lattices; Hecke operators on modular forms and their q-expansions; Hecke operators are Hermitian; multiplicity one.

Reading

1. F. Diamond and J. Shurman, *A First Course in Modular Forms,* Graduate Texts in Mathematics 228, Springer-Verlag, 2005.
2. R.C. Gunning, *Lectures on Modular Forms*, Annals of mathematical studies 48, Princeton University Press, 1962.
3. J.S. Milne, *Modular Functions and Modular Forms*: www.jmilne.org/math/CourseNotes/mf.html
4. J.-P. Serre, Chapter VII, *A Course in Arithmetic,* Graduate Texts in Mathematics 7, Springer-Verlag, 1973.

2.18 C3.7 Elliptic Curves — Prof. Victor Flynn — 16HT

Level: M-Level. **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

It is helpful, but not essential, if students have already taken a standard introduction to algebraic curves and algebraic number theory. For those students who may have gaps in their background, I have placed the file "Preliminary Reading" permanently on the Elliptic Curves webpage, which gives in detail (about 30 pages) the main prerequisite knowledge for the course. Go first to: http://www.maths.ox.ac.uk/courses/material then click on "C3.7 Elliptic Curves" and then click on the pdf file "Preliminary Reading".

Overview

Elliptic curves give the simplest examples of many of the most interesting phenomena which can occur in algebraic curves; they have an incredibly rich structure and have been the testing ground for many developments in algebraic geometry whilst the theory is still full of deep unsolved conjectures, some of which are amongst the oldest unsolved problems in mathematics. The course will concentrate on arithmetic aspects of elliptic curves defined over the rationals, with the study of the group of rational points, and explicit determination of the rank, being the primary focus. Using elliptic curves over the rationals as an example, we will be able to introduce many of the basic tools for studying arithmetic properties of algebraic varieties.

Learning Outcomes

On completing the course, students should be able to understand and use properties of elliptic curves, such as the group law, the torsion group of rational points, and 2-isogenies between elliptic curves. They should be able to understand and apply the theory of fields with valuations, emphasising the p-adic numbers, and be able to prove and apply Hensel's Lemma in problem solving. They should be able to understand the proof of the Mordell–Weil Theorem for the case when an elliptic curve has a rational point of order 2, and compute ranks in such cases, for examples where all homogeneous spaces for descent-via-2-isogeny satisfy the Hasse principle. They should also be able to apply the elliptic curve method for the factorisation of integers.

Synopsis

Non-singular cubics and the group law; Weierstrass equations.
Elliptic curves over finite fields; Hasse estimate (stated without proof).
p-adic fields (basic definitions and properties).
1-dimensional formal groups (basic definitions and properties).
Curves over p-adic fields and reduction mod p.
Computation of torsion groups over $\mathbb{Q}$; the Nagell–Lutz theorem.
2-isogenies on elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Weak Mordell–Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Height functions on Abelian groups and basic properties.
Heights of points on elliptic curves defined over $\mathbb{Q}$; statement (without proof) that this gives a height function on the Mordell–Weil group.
Mordell–Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Explicit computation of rank using descent via 2-isogeny.
Public keys in cryptography; Pollard's $(p-1)$ method and the elliptic curve method of factorisation.

Reading

1. J.W.S. Cassels, *Lectures on Elliptic Curves,* LMS Student Texts 24 (Cambridge University Press, 1991).

2. N. Koblitz, *A Course in Number Theory and Cryptography,* Graduate Texts in Mathematics 114 (Springer, 1987).

3. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves,* Undergraduate Texts in Mathematics (Springer, 1992).

4. J.H. Silverman, *The Arithmetic of Elliptic Curves,* Graduate Texts in Mathematics 106 (Springer, 1986).

Further Reading

1. A. Knapp, *Elliptic Curves, Mathematical Notes 40* (Princeton University Press, 1992).

2. G, Cornell, J.H. Silverman and G. Stevans (editors), *Modular Forms and Fermat's Last Theorem* (Springer, 1997).

3. J.H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves,* Graduate Texts in Mathematics 151 (Springer, 1994).

2.19 C3.8: Analytic Number Theory — Prof. Ben Green — 16MT

Level: M-Level. **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Basic ideas of complex analysis. Elementary number theory. Some familiarity with Fourier series will be helpful but not essential.

Overview

The aim of this course is to study the prime numbers using the famous Riemann ζ-function. In particular, we will study the connection between the primes and the zeros of the ζ-function. We will state the Riemann hypothesis, perhaps the most famous unsolved problem in mathematics, and examine its implication for the distribution of primes. We will prove the prime number theorem, which states that the number of primes less than X is asymptotic to $X/\log X$.

Learning Outcomes

In addition to the highlights mentioned above, students will gain experience with different types of Fourier transform and with the use of complex analysis.

Synopsis

Introductory material on primes. Arithmetic functions: Möbius function, Euler's ϕ-function, the divisor function, the σ-function. Multiplicativity. Dirichlet series and Euler products. The von Mangoldt function.

The Riemann ζ-function for $\Re(s) > 1$. Euler's proof of the infinitude of primes. ζ and the von Mongoldt function.

Schwarz functions on $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{R}/\mathbf{Z}$ and their Fourier transforms. *Inversion formulas and uniqueness*. The Poisson summation formula. The θ-function and its functional equation. The Γ-function and the meromorphic continuation and functional equation of the ζ-function. Poles and zeros of ζ and statement of the Riemann hypothesis. Basic estimates for ζ.

Product theorems for entire functions. Hadamard's product formula for ζ and the partial fraction expansion.

The Mellin transform of a smooth compactly supported function. The Mellin inversion formula. Decay of the Mellin transform in vertical strips. Statement and proof of the explicit formula.

The classical zero-free region. Proof of the prime number theorem. Implications of the Riemann hypothesis for the distribution of primes.

Reading

Full printed notes will be provided for the course, including the non-examinable topics (marked with asterisks above). The following books are relevant to the course.

1. G. H. Hardy and E. M. Wright, An introduction to the Theory of Numbers (Sixth edition, OUP 2008). Chapters 16, 17, 18.
2. H. Davenport, Multiplicative number theory (Third Edition, Springer Graduate texts 74), selected parts of the first half.
3. M. du Sautoy, Music of the primes (this is a popular book which could be useful background reading for the course).

2.20 C3.9: Computational Algebraic Topology — Prof Ulrike Tillmann & Prof Samson Abramsky 16HT

Level: M-level **Method of Assessment:** Mini-project (see section **??**)

Weight: Unit

Prerequisites

Some familiarity with the main concepts from algebraic topology, homological algebra and category theory will be helpful.

Overview

Ideas and tools from algebraic topology have become more and more important in computational and applied areas of mathematics. This course will provide at the masters level an introduction to the main concepts of (co)homology theory, and explore areas of applications in data analysis and in foundations of quantum mechanics and quantum information.

Learning outcomes

Students should gain a working knowledge of homology and cohomology of simplicial sets and sheaves, and improve their geometric intuition. Furthermore, they should gain an awareness of a variety of application in rather different, research active fields of applications with an emphasis on data analysis and contextuality.

Synopsis

The course has two parts. The first part will introduce students to the basic concepts and results of (co)homology, including sheaf cohomology. In the second part applied topics are introduced and explored.

Core: Homology and cohomology of chain complexes. Algorithmic computation of boundary maps (with a view of the classification theorm for finitely generated modules over a PID). Chain homotopy. Snake Lemma. Simplicial complexes. Other complexes (Delaunay, Cech). Mayer-Vietoris sequence. Poincare duality. Alexander duality. Acyclic carriers. Discrete Morse theory. (6 lectures)

Topic A: Persistent homology: barcodes and stability, applications todata analysis, generalisations. (4 lectures)

*Topic B:*Sheaf cohomology and applications to quantum non-locality and contextuality.Sheaf-theoretic representation of quantum non-locality and contextuality asobstructions to global sections. Cohomological characterizations and proofs of contextuality.(6 lectures)

Reading List

H. Edelsbrunner and J.L. Harer, *Computational Topology -An Introduction*,AMS (2010).

See also, U. Tillmann, Lecture notes for CAT 2012, in http://people.maths.ox.ac.uk/tillmann/CAT.html

Topic A:

G. Carlsson, *Topology and data*, Bulletin A.M.S.46 (2009), 255-308.

H. Edelsbrunner, J.L. Harer, *Persistent homology: A survey*, Contemporary Mathematics 452 A.M.S. (2008), 257-282.

S. Weinberger, *What is ... Persistent Homology?*, Notices A.M.S. 58 (2011), 36-39.

P. Bubenik, J. Scott, *Categorification of Persistent Homology*, Discrete Comput. Geom. (2014), 600–627.

Topic B:

S. Abramsky and Adam Brandenburger, The Sheaf-Theoretic Structure Of Non-Locality and Contextuality. In *New Journal of Physics*, 13(2011), 113036, 2011.

S. Abramsky and L. Hardy, Logical Bell Inequalities, Phys. Rev. A 85, 062114 (2012).

S. Abramsky, S. Mansfield and R. Soares Barbosa, The Cohomology of Non-Locality and Contextuality, in *Proceedings of Quantum Physics and Logic 2011*, Electronic Proceedings in Theoretical Computer Science, vol. 95, pages 1–15, 2012.

2.21 C4.1: Functional Analysis — Dr David Seifert — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part A Topology, Part B Banach Spaces and Hilbert Spaces

Overview

This course builds on B4.1 and B4.2, by extending the theory of Banach spaces and operators. As well as developing general methods that are useful in operator theory, we shall look in more detail at the structure and special properties of "classical" sequence spaces and function spaces.

Synopsis

Normed spaces and Banach spaces; dual spaces, subspaces, direct sums and completions; quotient spaces and quotient operators.

Baire's Category Theorem and its consequences (review).

Hahn–Banach extension and separation theorems; the bidual space and reflexivity.

Smoothness and uniform convexity of norms; classical Banach spaces and their duals.

Compact sets and compact operators. Ascoli's theorem.

Operators with closed range; Fredholm operators.

Weak and weak* topologies. The Banach–Alaoglu theorem and Goldstine's theorem. Weak compactness.

Schauder bases; examples in classical spaces. Gliding-hump arguments.

Reading

1. M. Fabian et al., *Functional Analysis and Infinite-Dimensional Geometry* (Canadian Math. Soc, Springer 2001)

2. N.L. Carothers, *A Short Course on Banach Space Theory* (LMS Student Text, Cambridge University Press 2004).

Further Reading

1. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (Springer 2011)

2.22 C4.2 Linear Operators — Prof. Charles Batty — 16HT

Level: M-level **Method of assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Essential: B4.1, B4.2. Useful: C4.1

Overview

Many of the linear operators that arise in mathematical physics and models from other sciences are not bounded operators. Typically they are defined on a dense subspace of a Banach or Hilbert space. They may be closed operators, but sometimes it is necessary to find the appropriate closed extension of the operator and the domain of the extension may be unclear. This course describes some of the theory of unbounded operators, particularly spectral properties of closed operators and ways to convert them into bounded operators.

Synopsis

Review of bounded operators and spectrum.

Unbounded operators; closed and closable operators; adjoints, spectrum.

Operators on Hilbert space; symmetric, self-adjoint, essentially self-adjoint. Spectral theorem and functional calculus. Quadratic forms, simple differential operators.

Semigroups of operators, generators, Hille-Yosida theorem, dissipative operators.

Reading

E.B. Davies, Linear operators and their spectra, CUP, 2007

P. Lax, Functional Analysis, Wiley, 2002

Further Reading

M. Reed & B. Simon, Methods of modern mathematical physics I,II, Academic Press, 1972, 1975

E.B. Davies, Differential operators and spectral theory, CUP, 1995

2.23 C4.8: Complex Analysis: Conformal Maps and Geometry — Prof. Dmitry Belyaev — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

The only necessary prerequisite is the basic complex analysis covered in Part A: analytic functions, Taylor series, contour integration, Cauchy theorem, and residues. Integration option is recommended but not necessary.

Overview

The aim of the course is to teach the principal techniques and methods of analytic and geometric function theory. This is a beautiful subject on its own right but it also have many applications in other areas of mathematics: potential theory, analytic number theory, probability. In the recent years the theory of Loewner equation became a crucial tool in the study of statistical physics lattice models.

This course is a continuation of the basic undergraduate complex analysis course but has much more geometric emphasis. Our main subject will be the theory of conformal maps, their analytical and geometrical properties.

Learning Outcomes

Students will have been introduced to ideas and techniques of geometric function theory that play important role and have a lot of applications in other areas of analysis. In particular, they will learn the proof of the Riemann mapping theorem and the concept of conformal invariants.

Synopsis

- Riemann mapping theorem. The main goal will be to prove Riemann's theorem which tells us that any non-trivial simply-connected domain can be conformally mapped onto the unit disc. This will be the key result for the entire course since it will allow as to connect the geometry of the domain with the analytical properties of the map which sends this domain to the unit disc. Within this section we will discuss
  - Maximum principle and Schwarz lemma, hyperbolic metric and Möbius transformations
  - Normal families, Hurwitz theorem
  - Proof of Riemann uniformization theorem
  - Constructive uniformization: Christoffel-Schwarz mappings and zipper algorithm (no proofs)
  - Uniformization for multiply-connected domains (sketch of the proof)

  - Applications: Dirichlet problem

- Theory of univalent functions. Univalent function is another term for one-to-one analytical map. We will be mostly interested in their boundary behaviour and how it is related to the geometry of the boundary. This section will cover
  - Area theorem and coefficient estimates
  - Koebe 1/4 theorem, distortion theorems
  - Conformal invariants: extremal length and its applications

Reading

1. L. Ahlfors, *Complex analysis*. This is a very good advanced textbook on Complex analysis. If you are a bit rusty on the basic complex analysis, then you might find everything you need (and a bit more) in Chapters 1–4. We will cover some of the material from chapters 5–6.
2. L. Ahlfors, *Conformal Invariants*. We will cover some topics from Chapters 1–6.
3. Ch. Pommerenke, *Univalent functions*. This book is mostly for further reading. We will discuss some of the results that are covered in Chapters 1,5,6, and 10.
4. Ch. Pommerenke, *Boundary behaviour of conformal maps*. This is an updated version of the previous book. We will be interested in Chapters 1,4, and 8.
5. P. Duren, *Univalent functions*. This is an excellent book about general theory of the univalent functions. We are mostly interested in the first three chapters.
6. G. Goluzin, *Geometric Theory of Functions of a Complex Variable*. This book contains vast amount of information about the geometric function theory. We will cover some of the results from the first four chapters.

2.24 C8.1: Stochastic Differential Equations — Prof. Harald Oberhauser — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part A integration, B8.1 Martingales Through Measure Theory and B8.2 Continuous Martingales and Stochastic Calculus, is expected.

Overview

Stochastic differential equations have been used extensively in many areas of application, including finance and social science as well as in physics, chemistry. This course develops the theory of Itô's calculus and stochastic differential equations.

Learning Outcomes

The student will have developed an appreciation of stochastic calculus as a tool that can be used for defining and understanding diffusive systems.

Synopsis

Recap on Brownian motion, quadratic variation, Ito's calculus: stochastic integrals with respect to local martingales, Ito's formula.

Lévy's characterisation of Brownian motion, exponent and Cameron-Martin martingales, exponential inequality, Burkholder-Davis-Gundy inequalities, Girsanov's Theorem, the Martingale Representation Theorem, Dambis-Dubins-Schwarz.

Local time, motion and Tanaka's formula.

Stochastic differential equations: strong and weak solutions, questions of existence and uniqueness, diffusion processes. Discussion of the one-dimensional case, a comparison theorem. Numerical schemes.

Conformal invariance of Brownian motion.

Reading

1. Prof Oberhauser's online notes:

2. M. Yor and D. Revaz, *Continuous Martingales and Brownian Motion* (Springer).

3. R. Durrett, *Stochastic Calculus* (CRC Press).

Further Reading

1. N. Ikeda & S. Watanabe, *Stochastic Differential Equations and Diffusion Processes* (North–Holland Publishing Company, 1989).

2. I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus,* Graduate Texts in Mathematics 113 (Springer-Verlag, 1988).

3. L. C. G. Rogers & D. Williams, *Diffusions, Markov Processes and Martingales Vol 1 (Foundations) and Vol 2 (Ito Calculus)* (Cambridge University Press, 1987 and 1994).

4. H. P. McKean, *Stochastic Integrals* (Academic Press, New York and London, 1969).

5. B. Oksendal, *Stochastic Differential Equations: An introduction with applications* (Universitext, Springer, 6th edition). Chapters II, III, IV, V, part of VI, Chapter VIII (F).

2.25 C8.3: Combinatorics — Prof. Alex Scott — 16MT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part B Graph Theory is helpful, but not required.

Overview

An important branch of discrete mathematics concerns properties of collections of subsets of a finite set. There are many beautiful and fundamental results, and there are still many basic open questions. The aim of the course is to introduce this very active area of mathematics, with many connections to other fields.

Learning Outcomes

The student will have developed an appreciation of the combinatorics of finite sets.

Synopsis

Chains and antichains. Sperner's Lemma. LYM inequality. Dilworth's Theorem.

Shadows. Kruskal-Katona Theorem.

Intersecting families. Erdos-Ko-Rado Theorem. Cross-intersecting families.

VC-dimension. Sauer-Shelah Theorem.

t-intersecting families. Fisher's Inequality. Frankl-Wilson Theorem. Application to Borsuk's Conjecture.

Combinatorial Nullstellensatz.

Reading

1. Bela Bollobás, *Combinatorics*, CUP, 1986.
2. Stasys Jukna, *Extremal Combinatorics*, Springer, 2007

2.26 C8.4: Probabilistic Combinatorics — Prof. Oliver Riordan — 16HT

Level: M-level **Method of Assessment:** Written examination.

Weight: Unit

Recommended Prerequisites

Part B Graph Theory and Part A Probability. C8.3 Combinatorics is not as essential prerequisite for this course, though it is a natural companion for it.

Overview

Probabilistic combinatorics is a very active field of mathematics, with connections to other areas such as computer science and statistical physics. Probabilistic methods are essential for the study of random discrete structures and for the analysis of algorithms, but they can also provide a powerful and beautiful approach for answering deterministic questions. The aim of this course is to introduce some fundamental probabilistic tools and present a few applications.

Learning Outcomes

The student will have developed an appreciation of probabilistic methods in discrete mathematics.

Synopsis

First-moment method, with applications to Ramsey numbers, and to graphs of high girth and high chromatic number.
Second-moment method, threshold functions for random graphs.
Lovász Local Lemma, with applications to two-colourings of hypergraphs, and to Ramsey numbers.
Chernoff bounds, concentration of measure, Janson's inequality.
Branching processes and the phase transition in random graphs.
Clique and chromatic numbers of random graphs.

Reading

1. N. Alon and J.H. Spencer, *The Probabilistic Method* (third edition, Wiley, 2008).

Further Reading

1. B. Bollobás, *Random Graphs* (second edition, Cambridge University Press, 2001).
2. M. Habib, C. McDiarmid, J. Ramirez-Alfonsin, B. Reed, ed., *Probabilistic Methods for Algorithmic Discrete Mathematics* (Springer, 1998).
3. S. Janson, T. Luczak and A. Rucinski, *Random Graphs* (John Wiley and Sons, 2000).
4. M. Mitzenmacher and E. Upfal, *Probability and Computing : Randomized Algorithms and Probabilistic Analysis* (Cambridge University Press, New York (NY), 2005).

5. M. Molloy and B. Reed, *Graph Colouring and the Probabilistic Method* (Springer, 2002).

6. R. Motwani and P. Raghavan, *Randomized Algorithms* (Cambridge University Press, 1995).

2.27 CCD : Dissertations on a Mathematical Topic

Level : M-level

Weight : Double-unit (10,000).

Students may offer a double-unit dissertation on a Mathematical topic for examination at Part C. A double-unit is equivalent to a 32-hour lecture course. Students will have approximately 8 hours of supervision for a double-unit dissertation distributed over Michaelmas and Hilary terms. In addition there are lectures on writing mathematics and using LaTeX in Michaelmas and Hilary terms. See the lecture list for details.

Students considering offering a dissertation should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics* available at:

https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects.

Application

Students must apply to the Mathematics Projects Committee for approval of their proposed topic in advance of beginning work on their dissertation. Proposals should be addressed to the Chairman of the Projects Committee, c/o Mrs Helen Lowe, Room S0.20, Mathematical Institute and are accepted from the end of Trinity Term. All proposals must be received before 12noon on Friday of Week 0 of Michaelmas Full Term. For CD dissertations candidates should take particular care to remember that the project must have substantial mathematical content. The application form is available at
https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects.
Once a title has been approved, it may only be changed by approval of the Chairman of the Projects Committee.

Assessment

Each project is blind double marked. The marks are reconciled through discussion between the two assessors, overseen by the examiners. Please see the *Guidance Notes on Extended Essays and Dissertations in Mathematics* for detailed marking criteria and class descriptors.

Submission

THREE copies of your dissertation, identified by your candidate number only, should be sent to the Chairman of Examiners, FHS of Mathematics Part C, Examination Schools, Oxford, to arrive no later than **12noon on Monday of week 10, Hilary Term 2017**.

An electronic copy of your dissertation should also be submitted via the Mathematical Institute website. Further details may be found in the *Guidance Notes on Extended Essays and Dissertations in Mathematics.*

2.28 Computer Science: Units

Students in Part C may take units drawn from Part C of the Honour School of Mathematics and Computing. For full details of these units see the Department of Computer Science's website (http://www.cs.ox.ac.uk/teaching/courses/)

Please note that these four courses will be examined by mini-project (as for MSc students). Mini-projects will be handed out to candidates on the last Monday or Friday of the term in which the subject is being taught, and you will have to hand it in to the Exam Schools by noon on Monday of Week 1 of the following term. The mini-project will be designed to be completed in about four to five days. It will include some questions that are more open-ended than those on a standard sit-down exam. The work you submit should be your own work, and include suitable references.

Please note that the Computer Science courses in Part C are 50% bigger than those in earlier years, i.e. for each Computer Science course in the 3rd year undergraduates are expected to undertake about 10 hours of study per week, but 4th year courses will each require about 15 hours a week of study. Lecturers are providing this extra work in a variety of ways, e.g. some will give 16 lectures with extra reading, classes and/or practicals, whereas others will be giving 24 lectures, and others still will be doing something in between. Students will need to look at each synopsis for details on this.

The Computer Science units available are as follows:

- CCS1 Categories, Proofs and Processes
- CCS2 Quantum Computer Science
- CCS3 Automata, Logics and Games
- CCS4 Advanced Machine Learning [2]

2.29 COD : Dissertations on a Topic related to Mathematics

Level : M-level

Weight : Double-unit (10,000 words).

Students may offer a double-unit dissertation on a Mathematically related topic for examination at Part C. For example, applications of mathematics to another field (eg Maths in

[2] The listed pre-requisites for this course are intended as useful, rather than strict requirements. Students uncertain about the background required for the course should email the lecturer.

Music), historical topics, topics concentrating on the analysis of statistical data, or topics concentrating on the production of computer-generated data are acceptable as topics for an OD dissertation. (Topics in mathematical education are not allowed.)

A double-unit is equivalent to a 32-hour lecture course. Students will have approximately 8 hours of supervision for a double-unit dissertation distributed over Michaelmas and Hilary terms. In addition there are lectures on writing mathematics and using LaTeX in Michaelmas and Hilary terms. See the lecture list for details.

Candidates considering offering a dissertation should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics* available at:

https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects.

Application

Students must apply to the Mathematics Projects Committee for approval of their proposed topic in advance of beginning work on their dissertation. Proposals should be addressed to the Chairman of the Projects Committee, c/o Mrs Helen Lowe, Room S0.20, Mathematical Institute and are accepted from the end of Trinity Term. All proposals must be received before 12noon on Friday of Week 0 of Michaelmas Full Term. The application form is available at https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects.

Once a title has been approved, it may only be changed by approval of the Chairman of the Projects Committee.

Assessment

Each project is blind double marked. The marks are reconciled through discussion between the two assessors, overseen by the examiners. Please see the *Guidance Notes on Extended Essays and Dissertations in Mathematics* for detailed marking criteria and class descriptors.

Submission

THREE copies of your dissertation,identified by your candidate number only, should be sent to the Chairman of Examiners, FHS of Mathematics Part C, Examination Schools, Oxford, to arrive no later than **12noon on Monday of week 10, Hilary Term 2017**. An electronic copy of your dissertation should also be submitted via the Mathematical Institute website. Further details may be found in the *Guidance Notes on Extended Essays and Dissertations in Mathematics.*

2.30 List of Mathematics Department units available only if special approval is granted

For details of these courses, and prerequisites for them, please consult the Supplement to the Mathematics Course Handbook, Syllabus and Synopses for Mathematics Part C 2016–2017, for examination in 2017, https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/handbooks-synopses

C4.3 Functional Analytic Methods for PDE's

C4.6 Fixed Point Methods for Nonlinear PDEs

C5.1 Solid Mechanics

C5.2 Elasticity and Plasticity

C5.3 Statistical Mechanics

C5.4 Networks

C5.5 Perturbation Methods

C5.6 Applied Complex Variables

C5.7 Topics in Fluid Mechanics

C5.9 Mathematical Mechanical Biology

C5.11 Mathematical Geoscience

C5.12 Mathematical Physiology

C6.1 Numerical Linear Algebra

C6.2 Continuous Optimization

C6.3 Approximation of Functions

C6.4 Finite Element Methods for Partial Differential Equations

C7.1 Theoretical Physics (double unit)

C7.5 General Relativity I

C7.6 General Relativity II

C8.2 Stochastic Analysis and PDEs