UNIVERSITY OF OXFORD
Mathematical Institute

HONOUR SCHOOL OF MATHEMATICS & PHILOSOPHY

SUPPLEMENT TO THE UNDERGRADUATE HANDBOOK – 2014 Matriculation

SYLLABUS AND SYNOPSES OF LECTURE COURSES

Part A
2015-16
For examination in 2016

These synopses can be found at:
https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/handbooks-synopses

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Supplement to the Handbook
Honour School of Mathematics & Philosophy
Syllabus and Synopses for Part A 2015–16
for examination in 2016

Contents

1 Foreword

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet should be addressed to the Academic Administrator in the Mathematical Institute.

[See the current edition of the *Examination Regulations* for the full regulations governing these examinations.]

In Part A each candidate shall be required to offer four written papers in Mathematics from the schedule of papers for Part A (given below). These must include A0, A2, and either two of A3, A4, A5, A8 or one of A3, A4, A5, A8 and ASO, making a total of 7.5 hours assessment.

At the end of the Part A examination a candidate will be awarded a 'University Standardised Mark' (USM) for each Mathematics paper in Part A. A weighted average of these USMs will be carried forward into the classification awarded at the end of the third year. In the calculation of any averages used to arrive at the final classification, USMs for A2 will have twice the weight of the USMs awarded for A0, the Long Options and Paper ASO.

1.1 The Schedule of Papers

Paper A0) - Linear Algebra

This paper will contain 3 questions set on the CORE material in Algebra 1 (Linear Algebra) for Part A of the FHS of Mathematics. The paper will be of $1\frac{1}{2}$ hours' duration. Candidates are expected to answer 2 questions. Each question is out of 25 marks.

Paper A2 - Metric Spaces and Complex Analysis

This paper will contain 6 questions set on the CORE material in Metric Spaces and Complex Analysis for Part A of the FHS of Mathematics. The paper will be of 3 hours' duration. Each question is out of 25 marks. Candidates are expected to answer 4 questions.

Papers A3, A4, A5 and A8

These papers will contain questions on the OPTIONAL subjects listed below. Each paper will be of $1\frac{1}{2}$ hours' duration. In each paper there will be 3 questions with candidates expected to answer 2 questions. Each question is out of 25 marks.

Paper ASO

This paper will contain one question on each of the nine SHORT OPTIONS listed below. The paper will be of $1\frac{1}{2}$ hours' duration. Candidates may submit answers to as many questions as they wish, of which the best 2 will count.

Optional Subjects: ***From the FHS of Mathematics Part A:***

Options:

A3 - Rings and Modules
A4 - Integration
A5 - Topology
A8 - Probability

ASO - Short Options:

Number Theory
Group Theory
Projective Geometry
Introduction to Manifolds
Integral Transforms
Calculus of Variations
Graph Theory
Special Relativity
Modelling in Mathematical Biology

Candidates may also, with the support of their college tutors, apply to the Joint Committee for Mathematics and Philosophy for approval of other Optional Subjects as listed for Part A of the Honour School of Mathematics.

1.2 Procedure for seeking approval of other options where this is required

You may, if you have the support of your Mathematics tutor, apply to the Chairman of the Joint Committee for Mathematics and Philosophy for approval of one or more other options from the list of Mathematics Department options for Part A. This list can be found in the Supplement to the Mathematics Course Handbook giving syllabuses and synopses for courses in Mathematics Part A and within this Supplement.

Applications for special approval must be made through the candidate's college and sent to the Chairman of the Joint Committee for Mathematics and Philosophy, c/o Academic Administrator, Mathematical Institute, to arrive by **Monday of Week 2 of Hilary Term**. Be sure to consult your college tutors if you are considering asking for approval to offer one of these other options.

Given that each of these other options, which are all in applied mathematics, presume facility with some or other results and techniques covered in first or second year core Mathematics courses not taken by Mathematics & Philosophy candidates, such applications will be exceptional.

1.3 Syllabus and Synopses

The **syllabus** details in this booklet are those referred to in the *Examination Regulations* and have been approved by the Mathematics Teaching Committee for examination in Trinity

Term 2016.

The **synopses** in this booklet give some additional detail, and show how the material is split between the different lecture courses. They also include details of recommended reading.

2 CORE MATERIAL

2.1 Syllabus

The examination syllabi of the two core papers A0 and A2 shall be the mathematical content of the synopses for the courses

- A0 - Linear Algebra
- A2 - Metric Spaces and Complex Analysis

as detailed below.

2.2 Synopses of Lectures

2.2.1 A0: Linear Algebra — Prof. Ulrike Tillmann — 16 lectures MT

Overview

The core of linear algebra comprises the theory of linear equations in many variables, the theory of matrices and determinants, and the theory of vector spaces and linear maps. All these topics were introduced in the Prelims course. Here they are developed further to provide the tools for applications in geometry, modern mechanics and theoretical physics, probability and statistics, functional analysis and, of course, algebra and number theory. Our aim is to provide a thorough treatment of some classical theory that describes the behaviour of linear maps on a finite-dimensional vector space to itself, both in the purely algebraic setting and in the situation where the vector space carries a metric derived from an inner product.

Learning Outcomes

Students will deepen their understanding of Linear Algebra. They will be able to define and obtain the minimal and characteristic polynomials of a linear map on a finite-dimensional vector space, and will understand and be able to prove the relationship between them; they will be able to prove and apply the Primary Decomposition Theorem, and the criterion for diagonalisability. They will have a good knowledge of inner product spaces, and be able to apply the Bessel and Cauchy–Schwarz inequalities; will be able to define and use the adjoint of a linear map on a finite-dimensional inner product space, and be able to prove and exploit the diagonalisability of a self-adjoint map.

Synopsis

Definition of an abstract vector space over an arbitrary field. Examples. Linear maps. [1]

Definition of a ring. Examples to include $\mathbb{Z}$, $F[x]$, $F[A]$ (where A is a matrix or linear map), $\mathrm{End}(V)$. Division algorithm and Bezout's Lemma in $F[x]$. Ring homomorphisms and isomorphisms. Examples. [2]

Characteristic polynomials and minimal polynomials. Coincidence of roots. [1]

Quotient vector spaces. The first isomorphism theorem for vector spaces and rank-nullity. Induced linear maps. Applications: Triangular form for matrices over $\mathbb{C}$. Cayley-Hamilton Theorem. [2]

Primary Decomposition Theorem. Diagonalizability and Triangularizability in terms of minimal polynomials. Proof of existence of Jordan canonical form over $\mathbb{C}$ (using primary decomposition and inductive proof of form for nilpotent linear maps). [3]

Dual spaces of finite-dimensional vector spaces. Dual bases. Dual of a linear map and description of matrix with respect to dual basis. Natural isomorphism between a finite-dimensional vector space and its second dual. Annihilators of subspaces, dimension formula. Isomorphism between U^* and V^*/U°. [3]

Recap on real inner product spaces. Definition of non-degenerate symmetric bilinear forms and description as isomorphism between V and V^*. Hermitian forms on complex vector spaces. Review of Gram-Schmidt. Orthogonal Complements. [1]

Adjoints for linear maps of inner product spaces. Uniqueness. Concrete construction via matrices [1]

Definition of orthogonal/unitary maps. Definition of the groups O_n, SO_n, U_n, SU_n. Diagonalizability of self-adjoint and unitary maps. [2]

Reading

Richard Kaye and Robert Wilson, *Linear Algebra* (OUP, 1998) ISBN 0-19-850237-0. Chapters 2–13. [Chapters 6, 7 are not entirely relevant to our syllabus, but are interesting.]

Further Reading

1. Paul R. Halmos, *Finite-dimensional Vector Spaces*, (Springer Verlag, Reprint 1993 of the 1956 second edition), ISBN 3-540-90093-4. §§1–15, 18, 32–51, 54–56, 59–67, 73, 74, 79. [Now over 50 years old, this idiosyncratic book is somewhat dated but it is a great classic, and well worth reading.]

2. Seymour Lipschutz and Marc Lipson, *Schaum's Outline of Linear Algebra* (3rd edition, McGraw Hill, 2000), ISBN 0-07-136200-2. [Many worked examples.]

3. C. W. Curtis, *Linear Algebra—an Introductory Approach*, (4th edition, Springer, reprinted 1994).

4. D. T. Finkbeiner, *Elements of Linear Algebra* (Freeman, 1972). [Out of print, but available in many libraries.]

2.2.2 A2: Metric Spaces and Complex Analysis — Prof. Kevin McGerty and Dr Richard Earl — 32 lectures MT

Overview

The theory of functions of a complex variable is a rewarding branch of mathematics to study at the undergraduate level with a good balance between general theory and examples. It occupies a central position in mathematics with links to analysis, algebra, number theory, potential theory, geometry, topology, and generates a number of powerful techniques (for example, evaluation of integrals) with applications in many aspects of both pure and applied mathematics, and other disciplines, particularly the physical sciences.

In these lectures we begin by introducing students to the language of topology before using it in the exposition of the theory of (holomorphic) functions of a complex variable. The central aim of the lectures is to present Cauchy's Theorem and its consequences, particularly series expansions of holomorphic functions, the calculus of residues and its applications.

The course concludes with an account of the conformal properties of holomorphic functions and applications to mapping regions.

Learning Outcomes

Students will have been introduced to point-set topology and will know the central importance of complex variables in analysis. They will have grasped a deeper understanding of differentiation and integration in this setting and will know the tools and results of complex analysis including Cauchy's Theorem, Cauchy's integral formula, Liouville's Theorem, Laurent's expansion and the theory of residues.

Synopsis

Metric Spaces (10 lectures)

Basic definitions: metric spaces, isometries, continuous functions ($\varepsilon - \delta$ definition), homeomorphisms, open sets, closed sets. Examples of metric spaces, including metrics derived from a norm on a real vector space, particularly l^1, l^2, l^∞ norms on $\mathbb{R}^n$, the sup norm on the bounded real-valued functions on a set, and on the bounded continuous real-valued functions on a metric space. The characterisation of continuity in terms of the pre-image of open sets or closed sets. The limit of a sequence of points in a metric space. A subset of a metric space inherits a metric. Discussion of open and closed sets in subspaces. The closure of a subset of a metric space. [3]

Completeness (but not completion). Completeness of the space of bounded real-valued functions on a set, equipped with the norm, and the completeness of the space of bounded continuous real-valued functions on a metric space, equipped with the metric. Lipschitz maps and contractions. Contraction Mapping Theorem. [2.5]

Connected metric spaces, path-connectedness. Closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected. [2]

Compactness. Heine-Borel theorem. The image of a compact set under a continuous map between metric spaces is compact. The equivalence of continuity and uniform continuity for functions on a compact metric space. Compact metric spaces are sequentially compact. Statement (but no proof) that sequentially compact metric spaces are compact. Compact metric spaces are complete. [2.5]

Complex Analysis (22 lectures)

Basic geometry and topology of the complex plane, including the equations of lines and circles. [1]

Complex differentiation. Holomorphic functions. Cauchy-Riemann equations (including $z, \bar{z}$ version). Real and imaginary parts of a holomorphic function are harmonic. [2]

Recap on power series and differentiation of power series. Exponential function and logarithm function. Fractional powers examples of multifunctions. The use of cuts as method of defining a branch of a multifunction. [3]

Path integration. Cauchy's Theorem. (Sketch of proof only students referred to various texts for proof.) Fundamental Theorem of Calculus in the path integral/holomorphic situation. [2]

Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem. [4]

Laurent's expansion. Classification of isolated singularities. Calculation of principal parts, particularly residues. [2]

Residue Theorem. Evaluation of integrals by the method of residues (straightforward examples only but to include the use of Jordan's Lemma and simple poles on contour of integration). [3]

Extended complex plane, Riemann sphere, stereographic projection. Möbius transformations acting on the extended complex plane. Möbius transformations take circlines to circlines. [2]

Conformal mappings. Riemann mapping theorem (no proof): Möbius transformations, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Joukowski's transformation). [3]

Reading

1. W. A. Sutherland, *Introduction to Metric and Topological Spaces* (Second Edition, OUP, 2009).

2. H. A. Priestley, *Introduction to Complex Analysis* (second edition, OUP, 2003).

Further Reading

1. L. Ahlfors, *Complex Analysis* (McGraw-Hill, 1979).

2. Reinhold Remmert, *Theory of Complex Functions* (Springer, 1989) (Graduate Texts in Mathematics 122).

3 OPTIONS

3.1 Syllabus

The examination syllabi of the four papers of options, A3, A4, A5 and A8, shall be the mathematical content of the synopses for the courses

- A3 - Rings and Modules
- A4 - Integration
- A5 - Topology
- A8 - Probability

as detailed below, and such other options from Mathematics Part A as are approved by the Joint Committee of Mathematics and Philosophy.

3.2 Synopses of Lectures

This section contains the lecture synopses associated with the four options papers, A3, A4, A5 and A8.

3.2.1 A3: Rings and Modules — Prof. Kevin McGerty — 16 lectures HT

Overview

The first abstract algebraic object which are normally studied are groups, which arise naturally from the study of symmetries. The focus of this course is on rings, which generalise the kind of algebraic structure possessed by the integers: a ring has two operations, addition and multiplication, which interact in the usual way. The course begins by studying the fundamental concepts of rings: what are maps between them, when are two rings isomorphic etc. much as was done for groups. As an application, we get a general procedure for building fields, generalising the way one constructs the complex numbers from the reals. We then begin to study the question of factorization in rings, and find a class of rings, known as Principal Ideal Domains, where any element can be written uniquely as a product of prime elements generalising the case of the integers. Finally, we study modules, which roughly means we study linear algebra over certain rings rather than fields. This turns out to have useful applications to ordinary linear algebra and to abelian groups.

Learning Outcomes

Students should become familiar with rings and fields, and understand the structure theory of modules over a Euclidean domain along with its implications. The material underpins many later courses in algebra and number theory, and thus should give students a good background for studying these more advanced topics.

Synopsis

Definition of rings (not necessarily commutative or with an identity) and examples: $\mathbb{Z}$, fields, polynomial rings (in more than one variable), matrix rings. [1]

Zero-divisors, integral domains. Units. The characteristic of a ring. Discussion of fields of fractions and their characterization (proofs non-examinable) [1]

Homomorphisms of rings. Quotient rings, ideals and the first isomorphism theorem and consequences, e.g. Chinese remainder theorem. Relation between ideals in R and R/I. [2]

Prime ideals and maximal ideals, relation to fields and integral domains. Examples of ideals. [1]

Euclidean Domains. Examples. Principal Ideal Domains. EDs are PIDs. Application of quotients to constructing fields by adjunction of elements; examples to include $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$ and some finite fields. Degree of a field extension, the tower law. [2]

Unique factorisation for PIDs. Gauss's Lemma and Eisenstein's Criterion for irreducibility. [2.5]

Modules: Definition and examples: vector spaces, abelian groups, vector spaces with an endomorphism. Submodules and quotient modules and direct sums. The first isomorphism theorem. [1.5]

Row and column operations on matrices over a ring. Equivalence of matrices and canonical forms of matrices over a Euclidean Domain. [1.5]

Free modules and presentations of finitely generated modules. Structure of finitely generated modules of a Euclidean domain. [2]

Application to rational canonical form for matrices, and structure of finitely generated Abelian groups. [1]

Reading

1. Michael Artin, *Algebra* (2nd ed. Pearson, (2010). (Excellent text covering everything in this course and much more besides).

2. Neils Lauritzen, *Concrete Abstract Algebra*, CUP (2003) (Excellent on groups, rings and fields, and covers topics in the Number Theory course also. Does not cover material on modules).

3. P. B. Bhattacharya, S. K. Jain, S. R. Nagpaul, *Basic Abstract Algebra*, CUP (1994) (Covers all of the basic algebra material most undergraduate courses have).

4. B. Hartley, T. O. Hawkes, Chapman and Hall, *Rings, Modules and Linear Algebra*. (Possibly out of print, but many library should have it. Relatively concise and covers all the material in the course).

3.2.2 A4: Integration — Prof. Zhongmin Qian — 16 lectures HT

Overview

The course will exhibit Lebesgue's theory of integration in which integrals can be assigned to a huge range of functions on the real line, thereby greatly extending the notion of integration presented in Mods. The theory will be developed in such a way that it can be easily extended to a wider framework, but measures other than Lebesgue's will only be lightly touched.

Operations such as passing limits, infinite sums, or derivatives, through integral signs, or reversing the order of double integrals, are often taken for granted in courses in applied mathematics. Actually, they can occasionally fail. Fortunately, there are powerful convergence and other theorems allowing such operations to be justified under conditions which are widely applicable. The course will display these theorems and a wide range of their applications.

This is a course in rigorous applications. Its principal aim is to develop understanding of the statements of the theorems and how to apply them carefully. Knowledge of technical proofs concerning the construction of Lebesgue measure will not be an essential part of the course, and only outlines will be presented in the lectures.

Learning Outcomes

Synopsis

Measure spaces. Outer measure, null set, measurable set. The Cantor set. Lebesgue measure on the real line. Counting measure. Probability measures. Construction of a non-measurable set (non-examinable). Measurable function, simple function, integrable function. Reconciliation with the integral introduced in Prelims.

A simple comparison theorem. Integrability of polynomial and exponential functions over suitable intervals. Changes of variable. Fatou's Lemma (proof not examinable). Monotone Convergence Theorem (proof not examinable). Dominated Convergence Theorem. Corollaries and applications of the Convergence Theorems (including term-by-term integration of series).

Theorems of Fubini and Tonelli (proofs not examinable). Differentiation under the integral sign. Change of variables.

Brief introduction to L^p spaces. Hölder and Minkowski inequalities (proof not examinable).

Reading

1. Lecture notes for the course.
2. M. Capinski & E. Kopp, *Measure, Integral and Probability* (Second Edition, Springer, 2004).
3. F. Jones, *Lebesgue Integration on Euclidean Space* (Second Edition, Jones & Bartlett, 2000).

Further Reading

1. D. S. Kurtz & C. W. Swartz, *Theories of Integration* (Series in Real Analysis Vol.9, World Scientific, 2004).
2. H. A. Priestley, *Introduction to Integration* (OUP 1997).

   [Useful for worked examples, although adopts a different approach to construction of the integral].
3. H. L. Royden, *Real Analysis* (various editions; 4th edition has P. Fitzpatrick as co author).
4. E. M. Stein & R. Shakarchi, *Real Analysis: Measure Theory, Integration and Hilbert Spaces* (Princeton Lectures in Analysis III, Princeton University Press,2005).

3.2.3 A5: Topology — Prof. Cornelia Drutu — 16 lectures HT

Overview

Topology is the study of 'spatial' objects. Many key topological concepts were introduced in the Metric Spaces course, such as the open subsets of a metric space, and the continuity of a map between metric spaces. More advanced concepts such as connectedness and compactness were also defined and studied. Unlike in a metric space, there is no notion of distance between points in a topological space. Instead, one keeps track only of the open subsets, but this is enough to define continuity, connectedness and compactness. By dispensing with a metric, the fundamentals of proofs are often clarified and placed in a more general setting.

In the first part of the course, these topological concepts are introduced and studied. In the second part of the course, simplicial complexes are defined; these are spaces that are obtained by gluing together triangles and their higher-dimensional analogues in a suitable way. This is a very general construction: many spaces admit a homeomorphism to a simplicial complex, which is known as a triangulation of the space. At the end of the course, the proof of one of the earliest and most famous theorems in topology is sketched. This is the classification of compact triangulated surfaces.

Learning Outcomes

By the end of the course, a student should be able to understand and construct abstract arguments about topological spaces. Their topological intuition should also be sufficiently well-developed to be able to reason about concrete topological spaces such as surfaces.

Synopsis

Axiomatic definition of an abstract topological space in terms of open sets. Basic definitions: closed sets, continuity, homeomorphism, convergent sequences, connectedness and comparison with the corresponding definitions for metric spaces. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, cofinite topology. The Hausdorff condition. Subspace topology. [3 lectures]

Accumulation points of sets. Closure of a set. Interior of a set. Continuity if and only if $f\left(\overline{A}\right) \subseteq \overline{f\left(A\right)}$. [2 lectures]

Basis of a topology. Product topology on a product of two spaces and continuity of projections. [2 lectures]

Compact topological spaces, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Product of two compact spaces is compact. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Equivalence of sequential compactness and abstract compactness in metric spaces. [3 lectures]

Quotient topology. Quotient maps. Characterisation of when quotient spaces are Hausdorff in terms of saturated sets. Examples, including the torus, Klein bottle and real projective plane. [3 lectures]

Abstract simplicial complexes and their topological realisation. A triangulation of a space. Any compact triangulated surface is homeomorphic to the sphere with g handles ($g \geqslant 0$) or the sphere with h cross-caps ($h \geqslant 1$). (No proof that these surfaces are not homeomorphic, but a brief informal discussion of Euler characteristic.) [3 lectures]

Reading

W. A. Sutherland, *Introduction to Metric and Topological Spaces* (Oxford University Press, 1975). Chapters 2-6, 8, 9.1-9.4.
(New edition to appear shortly.)

J. R. Munkres, *Topology, A First Course* (Prentice Hall, 1974), chapters 2, 3, 7.

Further Reading

B. Mendelson, *Introduction to Topology* (Allyn and Bacon, 1975). (cheap paperback edition available).

G. Buskes, A. Van Rooij, *Topological Spaces* (Springer, 1997).

N. Bourbaki, *General Topology* (Springer, 1998).

J. Dugundji, *Topology* (Allyn and Bacon, 1966), chapters 3, 4, 5, 6, 7, 9, 11. [Although out of print, available in some libraries.]

3.2.4 A8: Probability — Prof. James Martin — 16 lectures MT

Overview

The first half of the course takes further the probability theory that was developed in the first year. The aim is to build up a range of techniques that will be useful in dealing with mathematical models involving uncertainty. The second half of the course is concerned with Markov chains in discrete time and Poisson processes in one dimension, both with developing the relevant theory and giving examples of applications.

Learning Outcomes

Synopsis

Continuous random variables. Jointly continuous random variables, independence, conditioning, functions of one or more random variables, change of variables. Examples including some with later applications in statistics. Moment generating functions and applications. Statements of the continuity and uniqueness theorems for moment generating functions. Characteristic functions (definition only). Convergence in distribution and convergence in probability. Markov and Chebyshev inequalities. Weak law of large numbers and central limit theorem for independent identically distributed random variables. Statement of the strong law of large numbers. Discrete-time Markov chains: definition, transition matrix, n-step transition probabilities, communicating classes, absorption, irreducibility, periodicity, calculation of hitting probabilities and mean hitting times. Recurrence and transience. Invariant distributions, mean return time, positive recurrence, convergence to equilibrium (proof not examinable), ergodic theorem (proof not examinable). Random walks (including symmetric and asymmetric random walks on Z, and symmetric random walks on Z^d). Poisson processes in one dimension: exponential spacings, Poisson counts, thinning and superposition.

Reading

G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes* (3rd edition, OUP, 2001). Chapters 4, 6.1-6.5, 6.8.

R. Grimmett and D. R. Stirzaker, *One Thousand Exercises in Probability* (OUP, 2001).

G. R. Grimmett and D J A Welsh, *Probability: An Introduction* (OUP, 1986). Chapters 6, 7.4, 8, 11.1-11.3.

J. R. Norris, *Markov Chains* (CUP, 1997). Chapter 1.

D. R. Stirzaker, *Elementary Probability* (Second edition, CUP, 2003). Chapters 7-9 excluding 9.9.

3.3 List of Mathematics Department options available only if special approval is granted

For details of these courses, and prerequisites for them, please consult the Supplement to the Mathematics Course Handbook, Syllabus and Synopses for Mathematics Part A 2015–2016, for examination in 2016,
https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/handbooks-synopses

A6: Differential Equations 2

A7: Numerical Analysis

A9: Statistics

A10: Fluids and Waves

A11: Quantum Theory

4 SHORT OPTIONS

4.1 Syllabus

The examination syllabi of the short options paper ASO shall be the mathematical content of the synopses for the courses

- Number Theory
- Group Theory
- Projective Geometry
- Introduction to Manifolds
- Integral Transforms
- Calculus of Variations
- Graph Theory
- Special Relativity
- Modelling in Mathematical Biology

as detailed below.

4.2 Synopses of Lectures

This section contains the lecture synopses associated with the short options paper ASO.

4.2.1 Number Theory — Dr Jennifer Balakrishnan — 8 lectures TT

Overview

Number theory is one of the oldest parts of mathematics. For well over two thousand years it has attracted professional and amateur mathematicians alike. Although notoriously 'pure' it has turned out to have more and more applications as new subjects and new technologies have developed. Our aim in this course is to introduce students to some classical and important basic ideas of the subject.

Learning Outcomes

Students will learn some of the foundational results in the theory of numbers due to mathematicians such as Fermat, Euler and Gauss. They will also study a modern application of this ancient part of mathematics.

Synopsis

The ring of integers; congruences; ring of integers modulo n; the Chinese Remainder Theorem.

Wilson's Theorem; Fermat's Little Theorem for prime modulus; Euler's phi-function. Euler's generalisation of Fermat's Little Theorem to arbitrary modulus; primitive roots.

Quadratic residues modulo primes. Quadratic reciprocity.

Factorisation of large integers; basic version of the RSA encryption method.

Reading

Alan Baker, *A Concise Introduction to the Theory of Numbers* (Cambridge University Press, 1984) ISBN: 0521286549 Chapters 1,3,4.

David Burton, *Elementary Number Theory* (McGraw-Hill, 2001).

Dominic Welsh, *Codes and Cryptography*, (Oxford University Press, 1988), ISBN 0-19853-287-3. Chapter 11.

4.2.2 Group Theory — Dr Richard Earl — 8 lectures TT

Overview

This group theory course develops the theory begun in prelims, and this course will build on that. After recalling basic concepts, the focus will be on two circles of problems.

1. The concept of free group and its universal property allow to define and describe groups in terms of generators and relations.

2. The notion of composition series and the Jordan-Hölder Theorem explain how to see, for instance, finite groups as being put together from finitely many simple groups. This leads to the problem of finding and classifying finite simple groups. Conversely, it will be explained how to put together two given groups to get new ones.

Moreover, the concept of symmetry will be formulated in terms of group actions and applied to prove some group theoretic statements.

Learning Outcomes

Students will learn to construct and describe groups. They will learn basic properties of groups and get familiar with important classes of groups. They will understand the crucial concept of simple groups. They will get a better understanding of the notion of symmetry by using group actions.

Synopsis

Free groups. Uniqueness of reduced words and universal mapping property. Normal subgroups of free groups and generators and relations for groups. Examples. [2]

Review of the First Isomorphism Theorem and proof of Second and Third Isomorphism Theorems. Simple groups, statement that A_n is simple (proof for $n = 5$). Definition and proof of existence of composition series for finite groups. Statement of the Jordan-Hölder Theorem. Examples. The derived subgroup and solvable groups. [3]

Discussion of semi-direct products and extensions of groups. Examples. [1]

Sylow's three theorems. Applications including classification of groups of small order. [2]

Reading

1. Humphreys, J. F. *A Course in Group Theory*, Oxford, 1996
2. Armstrong, M. A. *Groups and Symmetry*, Springer-Verlag, 1988

4.2.3 Projective Geometry — Prof. Andrew Dancer — 8 lectures TT

Overview

Projective spaces provide a means of extending vector spaces by adding points at infinity. The resulting geometry is in some respects better-behaved than that of vector spaces, especially as regards intersection properties. Projective geometry is a good application of many concepts from linear algebra, such as bilinear forms and duality. It also provides an introduction to algebraic geometry proper, that is, the study of spaces defined by algebraic equations, as many such spaces are best viewed as living inside projective spaces.

Learning Outcomes

Students will be familiar with the idea of projective space and the linear geometry associated to it, including examples of duality and applications to Diophantine equations.

Synopsis

1-2: Projective Spaces (as $P(V)$ of a vector space V). Homogeneous Co-ordinates. Linear Subspaces.

3-4: Projective Transformations. General Position. Desargues Theorem. Cross-ratio.

5: Dual Spaces. Duality.

6-7: Symmetric Bilinear Forms. Conics. Singular conics, singular points. Projective equivalence of non-singular conics.

7-8: Correspondence between P^1 and a non-singular conic. Simple applications to Diophantine Equations.

Reading

1. N.J. Hitchin, *Maths Institute notes on Projective Geometry*, found at http://people.maths.ox.ac.uk/hitchin/hitchinnotes/hitchinnotes.html
2. M. Reid and B. Szendrői, *Geometry and topology*, Cambridge University Press, 2005 (Chapter 5).

4.2.4 Introduction to Manifolds — Prof. Andrew Dancer — 8 lectures TT

Overview

In this course, the notion of the total derivative for a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is introduced. Roughly speaking, this is an approximation of the function near each point in $\mathbb{R}^n$ by a linear transformation. This is a key concept which pervades much of mathematics, both pure and applied. It allows us to transfer results from linear theory locally to nonlinear functions. For example, the Inverse Function Theorem tells us that if the derivative is an invertible linear mapping at a point then the function is invertible in a neighbourhood of this point. Another example is the tangent space at a point of a surface in $\mathbb{R}^3$, which is the plane that locally approximates the surface best.

Learning Outcomes

Students will understand the concept of derivative in n dimensions and the implict and inverse function theorems which give a bridge between suitably nondegenerate infinitesimal information about mappings and local information. They will understand the concept of manifold and see some examples such as matrix groups.

Synopsis

Definition of a derivative of a function from R^m to R^n; examples; elementary properties; partial derivatives; the chain rule; the gradient of a function from R^n to R; Jacobian. Continuous partial derivatives imply differentiability, Mean Value Theorems. [3 lectures]

The Inverse Function Theorem and the Implicit Function Theorem (proofs non-examinable). [2 lectures]

The definition of a submanifold of R^m. Its tangent and normal space at a point, examples, including two-dimensional surfaces in R^3. [2 lectures]

Lagrange multipliers. [1 lecture]

Reading

Theodore Shifrin, *Multivariable Mathematics* (Wiley, 2005). Chapters 3-6.

T. M. Apostol, *Mathematical Analysis: Modern Approach to Advanced Calculus (World Students)* (Addison Wesley, 1975). Chapters 12 and 13.

S. Dineen, *Multivariate Calculus and Geometry* (Springer, 2001). Chapters 1-4.

J. J. Duistermaat and J A C Kolk, *Multidimensional Real Analysis I, Differentiation* (Cambridge University Press, 2004).

M. Spivak, *Calculus on Manifolds: A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965.

Further Reading

William R. Wade, *An Introduction to Analysis* (Second Edition, Prentice Hall, 2000). Chapter 11.

M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice Hall, 1976).

Stephen G. Krantz and Harold R. Parks, *The Implicit Function Theorem: History, Theory and Applications* (Birkhaeuser, 2002).

4.2.5 Integral Transforms — Dr Richard Earl — 8 lectures HT

This course is a prerequisite for Differential Equations 2.

Overview

The Laplace and Fourier Transforms aim to take a differential equation in a function f and turn it in an algebraic equation involving its transform $\bar{f}$ or $\hat{f}$. Such an equation can then be solved by algebraic manipulation, and the original solution determined by recognizing its transform or applying various inversion methods.

The Dirac δ-function, which is handled particularly well by transforms, is a means of rigorously dealing with ideas such as instantaneous impulse and point masses, which cannot be properly modelled using functions in the normal sense of the word. δ is an example of a *distribution* or *generalized function* and the course provides something of an introduction to these generalized functions and their calculus.

Learning Outcomes

Students will gain a range of techniques employing the Laplace and Fourier Transforms in the solution of ordinary and partial differential equations. They will also have an appreciation of generalized functions, their calculus and applications.

Synopsis

Motivation for a "function" with the properties the Dirac δ-function. Test functions. Continuous functions are determined by $\phi \to \int f\phi$. Distributions and δ as a distribution. Differentiating distributions. (3 lectures)

Theory of Fourier and Laplace transforms, inversion, convolution. Inversion of some standard Fourier and Laplace transforms via contour integration.

Use of Fourier and Laplace transforms in solving ordinary differential equations, with some examples including δ.

Use of Fourier and Laplace transforms in solving partial differential equations; in particular, use of Fourier transform in solving Laplace's equation and the Heat equation. (5 lectures)

Reading

P. J. Collins, *Differential and Integral Equations* (OUP, 2006), Chapter 14

W. E. Boyce & R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems* (7th edition, Wiley, 2000). Chapter 6

K.F. Riley & M. P. Hobson, *Essential Mathematical Methods for the Physical Sciences* (CUP 2011) Chapter 5

H.A. Priestley, *Introduction to Complex Analysis* (2nd edition, OUP, 2003) Chapters 21 and 22

Further Reading

L. Debnath & P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, (3rd Edition, Academic Press. 2005) Chapter 6

4.2.6 Calculus of Variations — Prof. Philip Maini — 8 lectures TT

Overview

The calculus of variations concerns problems in which one wishes to find the minima or extrema of some quantity over a system that has functional degrees of freedom. Many important problems arise in this way across pure and applied mathematics and physics. They range from the problem in geometry of finding the shape of a soap bubble, a surface that minimizes its surface area, to finding the configuration of a piece of elastic that minimises its energy. Perhaps most importantly, the principle of least action is now the standard way to formulate the laws of mechanics and basic physics.

In this course it is shown that such variational problems give rise to a system of differential equations, the Euler-Lagrange equations. Furthermore, the minimizing principle that underlies these equations leads to direct methods for analysing the solutions to these equations. These methods have far reaching applications and will help develop students technique.

Learning Outcomes

Students will be able to formulate variational problems and analyse them to deduce key properties of system behaviour.

Synopsis

The basic variational problem and Euler's equation. Examples, including axi-symmetric soap films.

Extension to several dependent variables. Hamilton's principle for free particles and particles subject to holonomic constraints. Equivalence with Newton's second law. Geodesics on surfaces. Extension to several independent variables.

Examples including Laplace's equation. Lagrange multipliers and variations subject to constraint. Eigenvalue problems for Sturm-Liouville equations. Legendre Polynomials.

Reading

Arfken Weber, *Mathematical Methods for Physicists* (5th edition, Academic Press, 2005). Chapter 17.

Further Reading

N. M. J. Woodhouse, *Introduction to Analytical Dynamics* (1987). Chapter 2 (in particular 2.6). (This is out of print, but still available in most College libraries.)

M. Lunn, *A First Course in Mechanics* (OUP, 1991). Chapters 8.1, 8.2.

P. J. Collins, *Differential and Integral Equations* (O.U.P., 2006). Chapters 11, 12.

4.2.7 Graph Theory — Prof. Peter Keevash — 8 lectures TT

Overview

This course introduces some central topics in graph theory.

Learning Outcomes

Students should have an appreciation of the flavour of methods and results in graph theory.

Synopsis

Introduction. Paths, walks and cycles. Trees and their characterisation, Cayley's theorem on counting trees. Euler circuits, Dirac's theorem on Hamilton cycles. Hall's theorem and matchings in bipartite graphs. Ramsey Theory.

Reading

R. J. Wilson, *Introduction to Graph Theory*, 5th edition, Prentice Hall, 2010.

D.B. West, *Introduction to Graph Theory*, 2nd edition, Prentice Hall, 2001.

4.2.8 Special Relativity — Prof. Lionel Mason — 8 lectures TT

Overview

The unification of space and time into a four-dimensional space-time is essential to the modern understanding of physics. This course will build on first-year algebra, geometry, and applied mathematics to show how this unification is achieved. The results will be illustrated throughout by reference to the observed physical properties of light and elementary particles.

Learning Outcomes

Students will be able to describe the fundamental phenomena of relativistic physics within the algebraic formalism of four-vectors. They will be able to solve simple problems involving Lorentz transformations. They will acquire a basic understanding of how the four-dimensional picture completes and supersedes the physical theories studied in first-year work.

Synopsis

Constancy of the speed of light. Lorentz transformations; time dilation, length contraction, the relativistic Doppler effect.

Index notation, four-vectors, four-velocity and four-momentum; equivalence of mass and energy; particle collisions and four-momentum conservation; equivalence of mass and energy: $E = mc^2$; four-acceleration and four-force, the example of the constant-acceleration world-line.

Reading

N. M. J. Woodhouse, *Special Relativity*, (Springer, 2002).

4.2.9 Modelling in Mathematical Biology — Prof. Ruth Baker — 8 lectures TT

Overview

Modelling in Mathematical Biology introduces the applied mathematician to practical applications in an area that is growing very rapidly. The course focuses on examples from population biology that can be analysed using deterministic discrete- and continuous-time non-spatial models, and demonstrates how mathematical techniques such as linear stability analysis and phase planes can enable us to predict the behaviour of living systems.

Learning Outcomes

Students will have developed a sound knowledge and appreciation of the ideas and concepts related to modelling biological and ecological systems using both discrete- and continuous-time non-spatial models.

Synopsis

Continuous population models for a single species, hysteresis and harvesting.

Discrete population models for a single species: oscillations, bifurcations and chaos.

Modelling interacting populations, including predator-prey and the principle of competitive exclusion.

Infectious disease modelling, including delays.

Discrete models for several species.

Reading

J. D. Murray, *Mathematical Biology*, Volume I: An Introduction. 3rd Edition, Springer (2002).

J. D. Murray, *Mathematical Biology*, Volume II: Spatial Models and Biomedical Applications. 3rd Edition, Springer (2003).

Further Reading

N. F. Britton, *Essential Mathematical Biology.* Springer (2003).

G. de Vries, T. Hillen, M. Lewis, J. Mller, B. Schnfisch. *A Course in Mathematical Biology: Quantitative Modelling with Mathematical and Computational Methods.* SIAM (2006).

5 Pathways to Part B

Most, but not all, third year options (Part B) have certain pre-requisites from Part A. Whilst the courses that will be offered in Part B in a year's time (to follow on from the Part A courses detailed in this supplement) have not been wholly decided there is not substantial change year-on-year in the list of Part B options offered.

What follows is a list envisaging how the Part A options for 2015-16 would be pre-requisites or useful knowledge for the Part B options for 2015-16. You will note that there are also a good number of courses that have no prerequisites.

- **A3 Rings and Modules**

  Essential for B2.1: Introduction to Representation Theory

  Essential for B2.2: Commutative Algebra

  Essential for B3.1: Galois Theory

  Essential for B3.4: Algebraic Number Theory

- **A4 Integration**

  Recommended for B4.1: Banach Spaces

  Recommended for B4.2: Hilbert Spaces

  Essential for B8.1: Martingales Through Measure Theory

  Essential for B8.2: Continuous Martingales and Stochastic Calculus

- **A5 Topology**

  Recommended for B3.2: Geometry of Surfaces

  Useful for B3.3: Algebraic Curves

  Essential for B3.5: Topology and Groups

- **A8 Probability**

  Useful for B5.1: Stochastic Modelling of Biological Processes

  Essential for B8.1: Martingales Through Measure Theory

  Essential for B8.2: Continuous Martingales and Stochastic Calculus

  Useful for B8.4: Communication Theory

  Essential for SB3a: Applied Probability

- **ASO: Number Theory**

  Useful for B3.1: Galois Theory

  Recommended for B3.4: Algebraic Number Theory

- **ASO: Group Theory**

  Recommended for B2.1: Introduction to Representation Theory

  Recommended for B3.1: Galois Theory

  Recommended for B3.4: Algebraic Number Theory

  Recommended for B3.5: Topology and Groups

- **ASO: Projective Geometry**

  Recommended for B3.3: Algebraic Curves

- **ASO: Introduction to Manifolds**

  Useful for B3.2: Geometry of Surfaces

  Useful for B3.3: Algebraic Curves

- **ASO: Integral Transforms**

  Useful for B4.2: Hilbert Spaces

- **ASO: Calculus of Variations** – no Part B courses open to M&P students explicitly require this.

- **ASO: Graph Theory**

  Recommended for B8.5: Graph Theory

- **ASO: Special Relativity** – no Part B courses explicitly require this.

- **ASO: Modelling in Mathematical Biology** – this course alone will not provide the necessary prerequisites to pursue this subject at Part B.

The following Part B courses in 2015-16 have no prerequisites.

- B1.1: Logic
- B1.2: Set Theory
- BEE/BOE Mathematical/Other Mathematical Extended Essay
- BN1.1 Mathematics Education
- BN1.2 Undergraduate Ambassadors' Scheme
- BO1.1: History of Mathematics
- OCS1 Lambda Calculus and Types
- OCS2 Computational Complexity
- OCS3 Knowledge Representation and Reasoning
- OCS4 Computer-aided Formal Verification