ADMISSIONS EXERCISE

MSc in Mathematical Finance

For entry January 2013

- *You should attempt all questions and show all working.*
- *Stating the answers without showing how they were obtained will not attract credit.*
- *You must sign and return the statement overleaf.*

Statement of authenticity

Please sign and return the following statement together with the solutions. Your application will not be considered without it.

I certify that the work I am submitting here is entirely my own and unaided work.

Print Name ____

Signed ____

Date ____

Linear algebra and calculus

1. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Prove the equivalence of the following statements.

i) $\mathbb{R}^3 = \ker(T) \oplus \mathrm{im}(T)$;

ii) $\ker(T) = \ker(T^2)$;

iii) $\mathrm{im}(T) = \mathrm{im}(T^2)$.

(We write $\mathbb{R}^3 = \ker(T) \oplus \mathrm{im}(T)$, if for all $v \in \mathbb{R}^3$ there exists $x \in \ker(T)$ and $y \in \mathrm{im}(T)$ such that $v = x + y$, and $\ker(T) \cap \mathrm{im}(T) = \{0\}$.)

(b) Magical squares are 3 by 3 matrices with the following properties: the sum of all numbers in each row, and in each column, and in each diagonal is equal. This number is called the magical number. For example

$$\begin{pmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{pmatrix}$$

is a magical square, and the magical number is 15.

i) Prove that the set of magical squares forms a vector space with the usual matrix addition and scalar-matrix product.

ii) Find a basis of the vector space of magical squares and determine its dimension.

2. (a) Let a function $f : (0,1] \to \mathbb{R}$ be given by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{p+q}$ if $x = p/q \in (0,1]$ in lowest term for integer p and q. Prove that this function is continuous at $\frac{1}{\sqrt{2}}$.

(b) Find a function $g : \mathbb{R} \to \mathbb{R}$ that is discontinuous at the points of the set $\{1/n : n \text{ is positive integer}\}$ but is continuous everywhere else, and whenever $x > y$, we also have $g(x) \geq g(y)$.

(c) Let $h : \mathbb{N} \times \mathbb{N} \to [0,1]$ be a function. Prove that

$$\sup\{\inf\{h(m,n) : m \in \mathbb{N}\} : n \in \mathbb{N}\} \leq \inf\{\sup\{h(m,n) : n \in \mathbb{N}\} : m \in \mathbb{N}\},$$

checking that each supremum and infimum exists.

(d) Let $f : (0,\infty) \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to\infty} f(x) = \ell,$$

for some $\ell \in \mathbb{R}$. Prove that

$$\lim_{X\to\infty} \frac{1}{X} \int_0^X f(x)\mathrm{d}x = \ell.$$

Differential equations

3. Let the function $c(S,t)$ satisfy the partial differential equation for $S \in (0,\infty)$ and $t \in [0,T]$

$$\frac{\partial c(S,t)}{\partial t} + rS\frac{\partial c(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} - rc(S,t) = 0,\ t \in (0,T),\ S \in (0,\infty) \quad (1)$$

with the terminal condition

$$c(S,T) = \max(S-K,0),$$

where $r, \sigma, K > 0$.

a) Let $F = Se^{r(T-t)}$ and $u(F,t) = c(S,t)$ for $t \in [0,T]$. Derive the partial differential equation and the terminal condition satisfied by $u(F,t)$.

b) Let X be positive, and define

$$v(S,t) = S^{\alpha} c\left(\frac{X^2}{S}, t\right),$$

where $\alpha = 1 - 2r/\sigma^2$. Prove that $v(S,t)$ satisfies the partial differential equation (**??**) that is satisfied by $c(S,t)$.

4. Use separation of variables to construct a series solution to the following problem

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < L,\ t > 0$$

$$u(0,t) = 0,\ u(L,t) = 0,\ t > 0$$

$$u(x,0) = \begin{cases} x & \text{if} \quad 0 < x < L/2 \\ L - x & \text{if} \quad L/2 < x < L \end{cases}$$

Probability

5. a) Two gamblers X and Y play a series of independent games. The probability that X wins a particular game is p and that Y wins is $q = 1 - p$. Each player begins with N units of money and in each game the winner collects one unit from the other. The series terminates when either X or Y loses all his money. We assume that $p > q$.

i) Derive the expected number of games in a series.

ii) Prove that the series terminates with probability 1.

iii) Let P_k denote the probability that X wins all Y's money starting from the position of X having $N + k$ units of money. Find P_k for $k = -N, \ldots, N$.

b) Let three boxes be labelled by A, B, and C. There are 3 white, 2 black and 1 red marbles in A, 2 white 1 black and 3 red marbles in B, and 1 white, 2 black and 2 red marbles in C. We randomly pick a box (each with probability 1/3) and then randomly draw two marbles from the chosen box.

Given that a red and a white marble have been drawn, what is the probability that the box B has been picked?

6. a) Let X and Y be independent gamma random variables with respective probability density functions

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \ x > 0, \text{ and } f_Y(y) = \frac{\lambda^\beta}{\Gamma(\beta)} y^{\beta-1} e^{-\lambda y}, \ y > 0,$$

where $\alpha, \beta, \lambda > 0$. Let $\eta = X/(X+Y)$ and $\nu = X + Y$.

i) Derive the joint density function of (η, ν).

ii) Prove that η and ν are independent and derive their density functions.

b) Let the random variable U be uniformly distributed on $(0, 1)$.

i) Find the distribution of the random variable

$$\xi = \frac{aU}{1-U}$$

where $a > 0$ is a constant. Does $\mathbb{E}[\xi]$ exist?

ii) Let $\hat{\xi}$ have the same distribution as ξ, and be independent of ξ. Derive the distribution of

$$\mu = \min\{\xi, \hat{\xi}\}.$$

Find $\mathbb{E}[\mu]$.

7. a) Let $X_1, X_2, \dots$ be a sequence of independent random variables such that

$$X_n = \begin{cases} \theta + n & \text{with probability} \quad 1/n \\ \theta & \text{with probability} \quad 1 - 1/n \end{cases}$$

Is this sequence convergent

i) in probability?

ii) in mean-square?

If it is convergent in either sense, specify the limit.

b) Let $(q_n)_{n>0}$ be a real sequence such that $0 < q_n < 1$ for all $n > 0$ and

$$\lim_{n\to\infty} q_n = 0.$$

For each $n > 0$, let X_n be a random variable, such that

$$\mathbb{P}[X_n = k] = q_n(1 - q_n)^{k-1}, \ (k = 1, 2, \dots).$$

Prove that the limit distribution of

$$\frac{X_n}{\mathbb{E}[X_n]}$$

is exponential with parameter 1.

Algorithms

8. The iterative sorting algorithm called `NaivRandomSort` works on arrays of distinct numbers as follows. Let A denote the array to be sorted. In each iteration, two indices i and j are randomly selected from uniform distribution on $\{1, \dots, n\}$ where n is the number of entries in A. If $A[\min(i, j)] > A[\max(i, j)]$, then the ith and jth entries are swapped. The algorithm stops when the array is sorted.

a) Prove that if A is not sorted, then the probability of a swap in the next iteration is at least $1/\binom{n}{2}$.

b) Prove that if A is not sorted, then the expected number of iterations until the next swap is at most $\binom{n}{2}$.

c) Prove that the algorithm stops after at most $\binom{n}{2}$ swaps performed by the algorithm.

[Hint: Explore how the number of pairs of indices (i, j) for which $A[\min(i, j)] > A[\max(i, j)]$ changes after each swap.]

d) What is the order of magnitude of the expected value of the total number of iterations?