Two-loop form factors: from $\mathcal{N} = 4$ SYM to QCD

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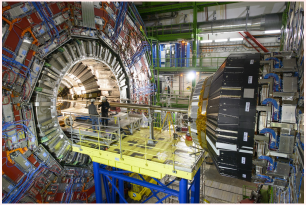
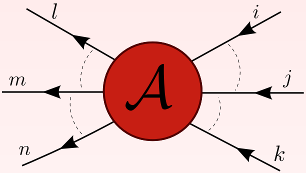
Outline

1. Intro
   - Motivations
   - Preliminaries
   - One-loop warm-up
2. Two-loops
   - Two-loop form factor
   - Two-loop remainder
   - Two-loop dilatation operator
3. Conclusions

Motivations: scattering amplitudes

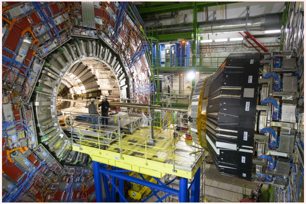


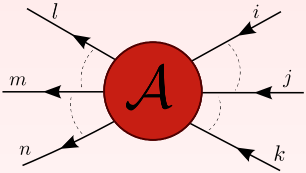


- Scattering amplitudes are natural observables in high-energy physics
- Testing ground for current theories of nature
- Exhibit striking simplicity and hidden symmetries
- E.g. for $gg \to n\,g$ scattering at **tree level**,

n	2	3	4	5	6	7	8
# diagrams	4	25	220	2485	34300	559405	10525900

Motivations: scattering amplitudes





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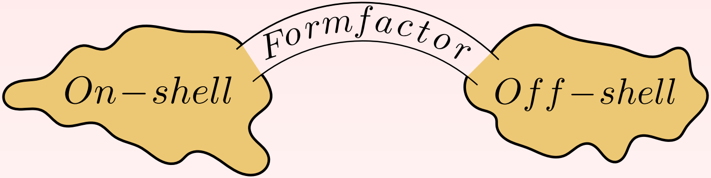
- Testing ground for current theories of nature
- Exhibit striking simplicity and hidden symmetries
- E.g. for $gg \to n\,g$ scattering at **tree level**, the result

$$\mathcal{A}(1^{\pm}, 2^{+}, \ldots, n^{+}) = 0$$

Motivations: form factors



- Study of form factors of local composite operators is an active area of research
- Partially off-shell:

$$F_{\mathcal{O}}(1,\ldots,n;q) = \int d^4x\, e^{iqx}\, \langle 1,\ldots,n|\mathcal{O}(x)|0\rangle$$

- We are interested in a particular case where $\mathcal{O}(x) = \mathrm{Tr}(F^3)$ and the external state $\langle g^+g^+g^+|$

Motivations: form factors

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e.g.: EM form factor, deep inelastic scattering, Mott scattering

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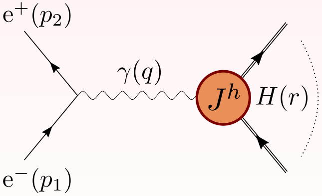
e.g.: EM form factor, deep inelastic scattering, **Mott scattering**
Probing the sub-structure of hadrons (protons) using electrons:



$$= \bar{v}(p_2)(ie\gamma_\mu)u(p_1)\,\frac{\eta^{\mu\nu}}{q^2}\,\langle r|J^h_\nu(0)|0\rangle$$

hardonic EM current

Motivations: form factors and amplitudes

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Motivations: form factors and amplitudes

$$F_{\mathcal{O}}(1,\ldots,n;q=0)=\int d^4x\;\langle 1,\ldots,n|\mathcal{O}(x)|0\rangle$$

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Correction to the **amplitude** from addition of a new coupling to the action:

$$\delta S = g_{\mathcal{O}}\int d^4x\,\mathcal{O}(x) \qquad \mathcal{A}=\langle 1,\ldots,n|0\rangle$$

$$\delta\mathcal{A} = g_{\mathcal{O}}\int d^4x\,\langle 1,\ldots,n|\mathcal{O}(x)|0\rangle+\mathcal{O}(g_{\mathcal{O}}^2)$$

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e.g. **QCD effective Lagrangian:** [Dawson, Lewis, Zeng]

$$\mathcal{L}_{\rm eff} = \hat{C}_1 O_1 + \frac{1}{m_{\rm top}^2}\sum_{i=2}^{5}\hat{C}_i O_i + \mathcal{O}(m_{\rm top}^{-4}) \qquad O_1 = H\mathrm{Tr}(F^2)$$

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Higgs amplitudes $\leftrightarrow$ form factors of Tr(F^2)!

Motivations: from $\mathcal{N} = 4$ SYM to QCD

$\mathcal{N} = 4$: two-loop form factor of $\mathrm{Tr}(\phi^2)$ equivalent to $\mathrm{Tr}(F^2)$:

[Brandhuber, Gurdogan, Mooney, Travaglini, Yang]

- $\mathrm{Tr}(F^2_{\mathrm{SD}})$ and $\mathrm{Tr}(\phi^2)$ are in the same protected stress-tensor multiplet
- Supersymmetric Ward identities relate the two form factors

$\mathcal{N} = 4$ two loop form factor of $\mathrm{Tr}(\phi^2)$ **identical** to maximally transcendental part of amplitudes for $H \to g^+ g^+ g^\pm$ in **QCD**!

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want to know $\longleftrightarrow$ able to compute

QCD		$\mathcal{N}=4$		$\mathcal{N}=4$
$\mathrm{Tr}(F^2)$	$\longleftrightarrow$	$\mathrm{Tr}(F^2)$	$\longleftrightarrow$	$\mathrm{Tr}(\phi^2)$
	same max. transcendental part		same supermultiplet	

Motivations

- **End goal**: be able to make statements about $\mathcal{O}_3 = \mathrm{Tr}(F^3)$ in QCD
- **For now**: study $\mathcal{N} = 4$ SYM length-3 operators built out of scalars:

$$\mathcal{O}_B = \mathrm{Tr}(\phi_{12}[\phi_{23}, \phi_{31}])\,, \qquad \tilde{\mathcal{O}}_{\mathrm{BPS}} = \mathrm{Tr}(\phi_{12}\{\phi_{23}, \phi_{31}\})$$

- $\mathcal{O}_B$ and $\mathrm{Tr}(F^3)$ have the same **one-loop anomalous dimension**

Anomalous dimension

In a CFT (e.g. $\mathcal{N} = 4$ SYM): no mass spectrum. Analogous notion is a **conformal dimension**: tells us how operators transform under dilatations

$$\mathcal{O}_\Delta(x) \to \lambda^{-\Delta}\mathcal{O}_\Delta(\lambda x)$$

$$\langle \mathcal{O}_\Delta(x)\bar{\mathcal{O}}_\Delta(y)\rangle = \frac{1}{|x-y|^{2\Delta}}$$

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γ is the **one-loop anomalous dimension**.

Motivations: form factors and anomalous dimension

- $\mathrm{Tr}(F^3)$ and $\mathrm{Tr}(X[Y,Z])$ have the same γ - **so what?**
- Expand:

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- $\mathrm{Tr}(F^3)$ and $\mathrm{Tr}(X[Y,Z])$ have the same γ - **so what?**
- Expand:

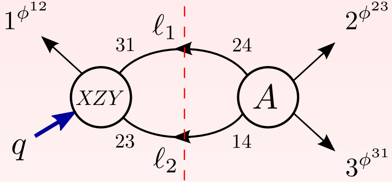
$$\langle \mathcal{O}_\Delta(x)\bar{\mathcal{O}}_\Delta(0)\rangle = \frac{1}{|x|^{2\Delta_0}}\left[1-\gamma\log(|x|^2\Lambda^2)+\ldots\right]$$

- γ is the coefficient of the UV divergence in the result of 1-loop form factor! [Zwiebiel, Wilhelm]
- Form factors of $\mathrm{Tr}(F^3)$ and $\mathrm{Tr}(X[Y,Z])$ have a chance to be related
- **Strategy**: focus on $\mathrm{Tr}(X[Y,Z])$ at two loops, simple(r). Learn and apply to $\mathrm{Tr}(F^3)$ - **many interesting lessons ahead!**

Preliminaries: $\mathcal{N} = 4$ SYM and mixing

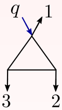
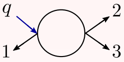
- Maximally supersymmetric theory in 4d: 2 gluons $G^{+/-}$, 8 fermions ψ_{ABC}, $\bar{\psi}_A$, 6 scalars ϕ_{AB} $A = 1, \ldots, 4$
- We consider **closed sectors**:
  - $SU(2|3)$ consisting of $\{\phi_{12}, \phi_{23}, \phi_{31}, \psi_{123}\}$
  - $SU(2)$ consisting of $\{\phi_{12}, \phi_{23}\}$ **(subsector!)**
- Closed in the sense of **operator mixing** - under renormalization, $\mathcal{O}^{REN} \sim \mathcal{O}^{BARE}$ but also other operators build out of letters forming the sector - **but not any other!**
- Our $\mathrm{Tr}(X[Y, Z])$ drags along a friend - $\mathrm{Tr}(\psi\psi)$
- Could also imagine another dim 3 operator - $\mathrm{Tr}(X\{Y, Z\})$
- This will turn out to be **half-BPS** - doesn't mix!

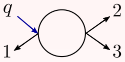
One-loop warm-up

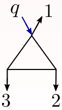
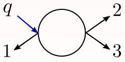


Generalised unitarity

Unitarity: reconstruct loop-level amplitudes from discontinuities calculated via "cuts"
Generalised: more general cuts that still lead to factorisation into lower-loop and tree-level

$F^{(1)}_{XYZ} = -\, i^3\, s_{23} \times$  $-\, i^3$ 

$F^{(1)}_{XZY} = i^3$ 

$F^{(1)}_{X[Y,Z]} = -i^3\, s_{23} \times$  $-\, 2\, i^3$ 

Two-loop cuts: useful trick

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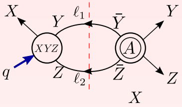

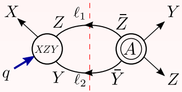

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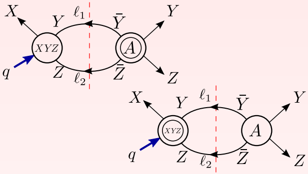

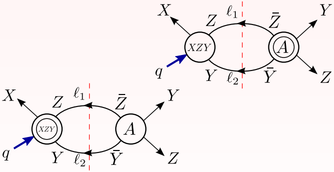

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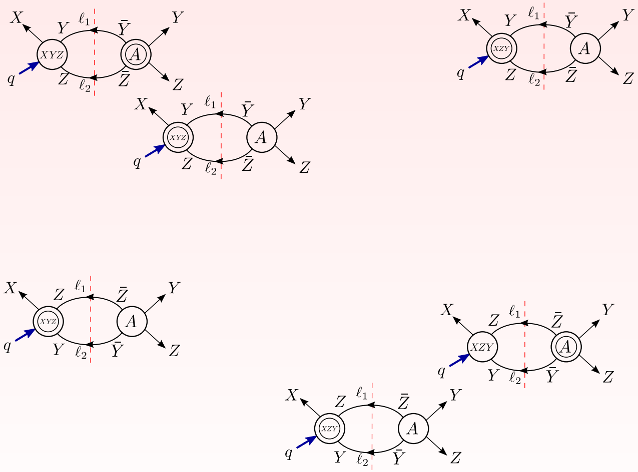

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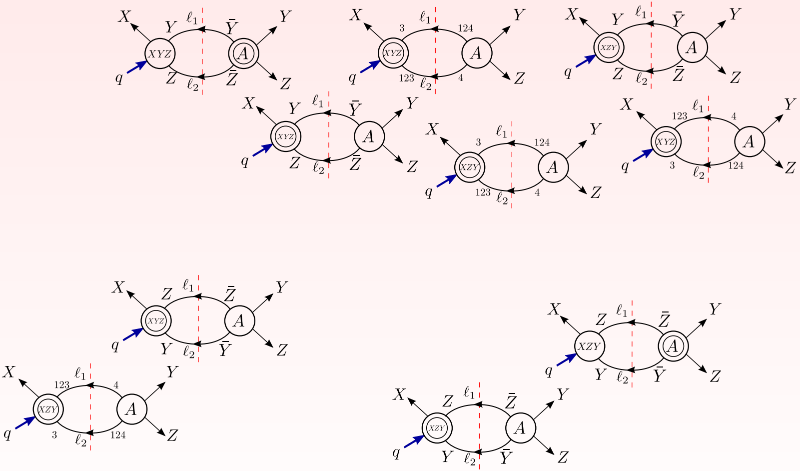

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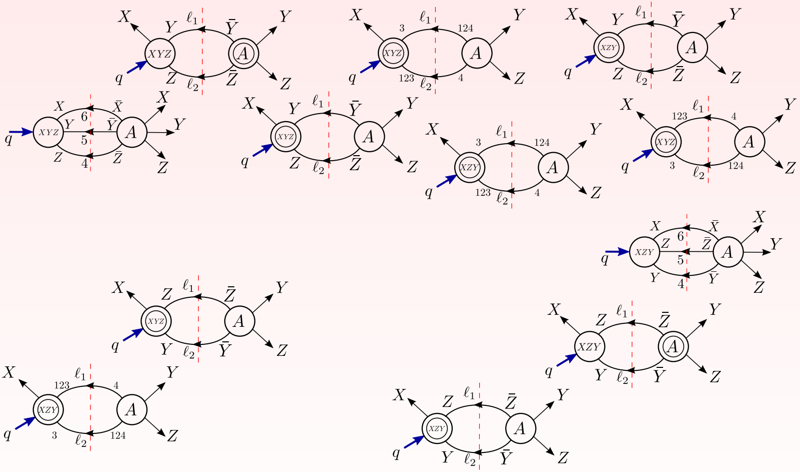

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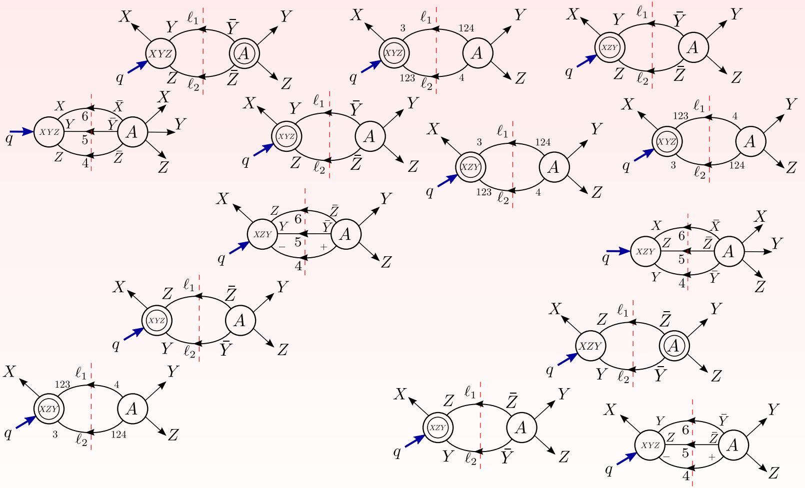

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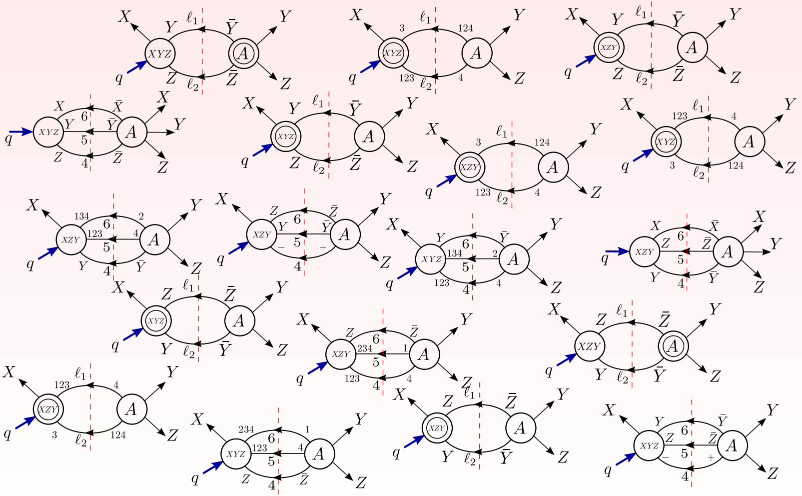

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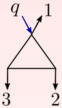
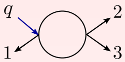
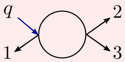
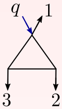
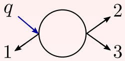
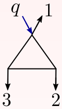
$F^{(1)}_{XYZ} = -\, s_{23} \times$  $-$ 

$F^{(1)}_{XZY} =$ 

$F^{(1)}_{\mathcal{O}_B} = -\, s_{23} \times$  $- \, 2$ 

$F^{(1)}_{X\{Y,Z\}} = -\, s_{23} \times$ 

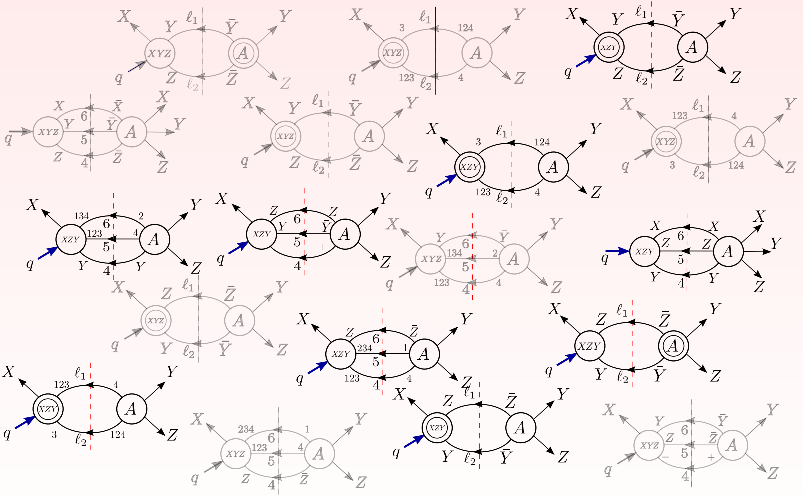
half-BPS!

Key observation

$\mathcal{O}_B = \mathcal{O}_{\rm BPS} - 2\mathrm{Tr}(XZY)$, useful as $F^{(2)}_{\mathcal{O}_{BPS}}$ done - same as $F^{(2)}_{\mathrm{Tr}X^3}$

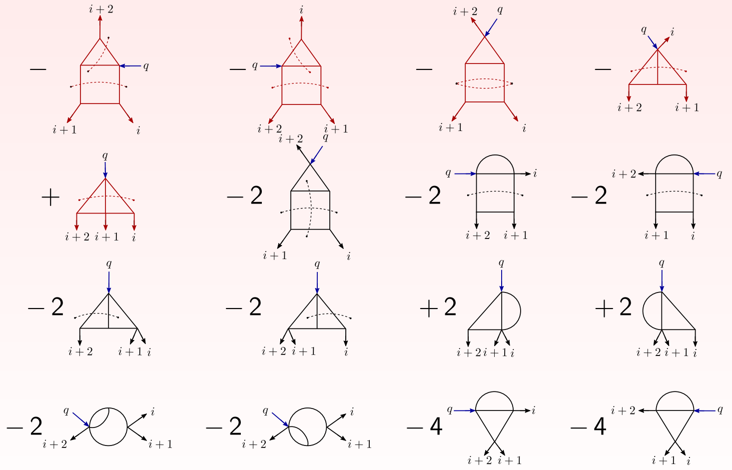
[Brandhuber, Penante, Travaglini, Wen]

Two-loop cuts: useful trick



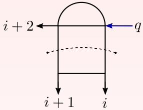
Two-loop cuts: result

$F^{(2)}_{\mathcal{O}_B} =$



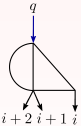
Two-loop reduction

Some of the integrals have known expressions and we can substitute immediately, e.g: [Gehrmann, Remiddi]

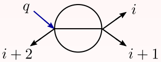


$$= \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (-q)^{-2\epsilon} \sum_{i=-1}^{3} \frac{f_i\left(\frac{s_{i+1\,i+2}}{q^2}, \frac{s_{1\,i+1}}{q^2}\right)}{\epsilon^i} + \mathcal{O}(\epsilon^2)$$

Others, we need to **reduce** using `LiteRed` algorithm, e.g: [Lee]



$$= \frac{3\epsilon - 2}{2\epsilon(s_{i\,i+2} + s_{i+1\,i+2})} \Bigg($$



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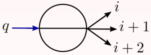

$$\Bigg)$$

Two-loop remainder

- Efficient way to present two-loop form factors
- Free of infrared divergences, helicity-blind
- Expressed in terms of at most transcendentality-four functions
- Rescaling invariant, depends on Mandelstam variables only through their ratios
- Defined in terms of two important universal constants: cusp anomalous dimension and collinear anomalous dimension:

$$\mathcal{R}^{(2)}_{\mathcal{O}} := F^{(2)}_{\mathcal{O}}(\epsilon) - \frac{1}{2}\big(F^{(1)}_{\mathcal{O}}(\epsilon)\big)^2 - f^{(2)}(\epsilon)\, F^{(1)}_{\mathcal{O}}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

$$f^{(2)}(\epsilon) := -2(\zeta_2 + \epsilon\,\zeta_3 + \epsilon^2\,\zeta_4)$$

[Bern, Dixon, Smirnov]
[Brandhuber, Travaglini, Yang]

Symbol

Result seemingly simple, but after substituting for the integrals...

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Result seemingly simple, but after substituting for the integrals...

$$5\,\text{HH}\left[0,\frac{s12}{s123}\right]-\text{HH}\left[0,0,\frac{s12}{s123}\right]+\text{HH}\left[1,0,\frac{s12}{s123}\right]+\epsilon\left(-\frac{65}{2}+19\,\text{HH}\left[0,\frac{s12}{s123}\right]+\frac{1}{6}\pi^2\left(5-\text{HH}\left[0,\frac{s12}{s123}\right]+\text{HH}\left[1,\frac{s12}{s123}\right]\right)-5\,\text{HH}\left[0,0,\frac{s12}{s123}\right]+5\,\text{HH}\left[1,0,\frac{s12}{s123}\right]+\right.$$

$$\text{HH}\left[0,0,0,\frac{s12}{s123}\right]-\text{HH}\left[0,1,0,\frac{s12}{s123}\right]-\text{HH}\left[1,0,0,\frac{s12}{s123}\right]+\text{HH}\left[1,1,0,\frac{s12}{s123}\right]+\text{zzeta}[3]\Big)+\epsilon^2\left(-\frac{211}{2}+\frac{3\pi^4}{40}+65\,\text{HH}\left[0,\frac{s12}{s123}\right]-19\,\text{HH}\left[0,0,\frac{s12}{s123}\right]+\right.$$

$$19\,\text{HH}\left[1,0,\frac{s12}{s123}\right]+\frac{1}{6}\pi^2\left(19-5\,\text{HH}\left[0,\frac{s12}{s123}\right]+5\,\text{HH}\left[1,\frac{s12}{s123}\right]+\text{HH}\left[0,0,\frac{s12}{s123}\right]-\text{HH}\left[0,1,\frac{s12}{s123}\right]-\text{HH}\left[1,0,\frac{s12}{s123}\right]+\text{HH}\left[1,1,\frac{s12}{s123}\right]\right)+5\,\text{HH}\left[0,0,0,\frac{s12}{s123}\right]-$$

$$5\,\text{HH}\left[0,1,0,\frac{s12}{s123}\right]-5\,\text{HH}\left[1,0,0,\frac{s12}{s123}\right]+5\,\text{HH}\left[1,1,0,\frac{s12}{s123}\right]-\text{HH}\left[0,0,0,0,\frac{s12}{s123}\right]+\text{HH}\left[0,0,1,0,\frac{s12}{s123}\right]+\text{HH}\left[0,1,0,0,\frac{s12}{s123}\right]-\text{HH}\left[0,1,1,0,\frac{s12}{s123}\right]+$$

$$\left.\left.\text{HH}\left[1,0,0,0,\frac{s12}{s123}\right]-\text{HH}\left[1,0,1,0,\frac{s12}{s123}\right]-\text{HH}\left[1,1,0,0,\frac{s12}{s123}\right]+\text{HH}\left[1,1,1,0,\frac{s12}{s123}\right]+\left(5-4\,\text{HH}\left[0,\frac{s12}{s123}\right]-2\,\text{HH}\left[1,\frac{s12}{s123}\right]\right)\text{zzeta}[3]\right)\right)-$$

$$\frac{1}{\left(\frac{s23}{2}+\frac{s31}{2}\right)\epsilon^2\,\text{Gamma}[1-2\epsilon]^2}\,2^{-9+4\epsilon}\,\pi^{-4+2\epsilon}\,(-s123)^{-2\epsilon}\,(1-2\epsilon)\,\text{Gamma}[1-\epsilon]^4\,\text{Gamma}[1+\epsilon]^2$$

$$(-8\,s23-8\,s31+2\,s23\,(4-2\epsilon)+2\,s31\,(4-2\epsilon)+10\,(p1\cdot p1)-3\,(4-2\epsilon)\,(p1\cdot p1)+10\,(p2\cdot p2)-3\,(4-2\epsilon)\,(p2\cdot p2)+10\,(p3\cdot p3)-3\,(4-2\epsilon)\,(p3\cdot p3))$$

$$\left(-\frac{19}{2}+\frac{\pi^2}{6}-\frac{1}{2\epsilon^2}+\frac{-\frac{5}{2}+\text{HH}\left[0,\frac{s12}{s123}\right]}{\epsilon}+5\,\text{HH}\left[0,\frac{s12}{s123}\right]-\text{HH}\left[0,0,\frac{s12}{s123}\right]+\text{HH}\left[1,0,\frac{s12}{s123}\right]+\right.$$

$$\epsilon\left(-\frac{65}{2}+19\,\text{HH}\left[0,\frac{s12}{s123}\right]+\frac{1}{6}\pi^2\left(5-\text{HH}\left[0,\frac{s12}{s123}\right]+\text{HH}\left[1,\frac{s12}{s123}\right]\right)-5\,\text{HH}\left[0,0,\frac{s12}{s123}\right]+5\,\text{HH}\left[1,0,\frac{s12}{s123}\right]+\text{HH}\left[0,0,0,\frac{s12}{s123}\right]-\text{HH}\left[0,1,0,\frac{s12}{s123}\right]-\right.$$

$$\text{HH}\left[1,0,0,\frac{s12}{s123}\right]+\text{HH}\left[1,1,0,\frac{s12}{s123}\right]+\text{zzeta}[3]\Big)+\epsilon^2\left(-\frac{211}{2}+\frac{3\pi^4}{40}+65\,\text{HH}\left[0,\frac{s12}{s123}\right]-19\,\text{HH}\left[0,0,\frac{s12}{s123}\right]+19\,\text{HH}\left[1,0,\frac{s12}{s123}\right]+\right.$$

$$\frac{1}{6}\pi^2\left(19-5\,\text{HH}\left[0,\frac{s12}{s123}\right]+5\,\text{HH}\left[1,\frac{s12}{s123}\right]+\text{HH}\left[0,0,\frac{s12}{s123}\right]-\text{HH}\left[0,1,\frac{s12}{s123}\right]-\text{HH}\left[1,0,\frac{s12}{s123}\right]+\text{HH}\left[1,1,\frac{s12}{s123}\right]\right)+5\,\text{HH}\left[0,0,0,\frac{s12}{s123}\right]-5\,\text{HH}\left[0,1,0,\frac{s12}{s123}\right]-$$

$$5\,\text{HH}\left[1,0,0,\frac{s12}{s123}\right]+5\,\text{HH}\left[1,1,0,\frac{s12}{s123}\right]-\text{HH}\left[0,0,0,0,\frac{s12}{s123}\right]+\text{HH}\left[0,0,1,0,\frac{s12}{s123}\right]+\text{HH}\left[0,1,0,0,\frac{s12}{s123}\right]-\text{HH}\left[0,1,1,0,\frac{s12}{s123}\right]+$$

$$\left.\left.\text{HH}\left[1,0,0,0,\frac{s12}{s123}\right]-\text{HH}\left[1,0,1,0,\frac{s12}{s123}\right]-\text{HH}\left[1,1,0,0,\frac{s12}{s123}\right]+\text{HH}\left[1,1,1,0,\frac{s12}{s123}\right]+\left(5-4\,\text{HH}\left[0,\frac{s12}{s123}\right]-2\,\text{HH}\left[1,\frac{s12}{s123}\right]\right)\text{zzeta}[3]\right)\right)+$$

$$\frac{1}{\text{Gamma}[1-2\epsilon]^2}\,2^{-5+4\epsilon}\,\pi^{-4+2\epsilon}\,(-s123)^{-2\epsilon}\,\text{Gamma}[1-\epsilon]^4\,\text{Gamma}[1+\epsilon]^2\left(-\frac{19}{2}+\frac{\pi^2}{6}-\frac{1}{2\epsilon^2}+\frac{-\frac{5}{2}+\text{HH}\left[0,\frac{s23}{s123}\right]}{\epsilon}+5\,\text{HH}\left[0,\frac{s23}{s123}\right]-\text{HH}\left[0,0,\frac{s23}{s123}\right]+\right.$$

$$\text{HH}\left[1,0,\frac{s23}{s123}\right]+\epsilon\left(-\frac{65}{2}+19\,\text{HH}\left[0,\frac{s23}{s123}\right]+\frac{1}{6}\pi^2\left(5-\text{HH}\left[0,\frac{s23}{s123}\right]+\text{HH}\left[1,\frac{s23}{s123}\right]\right)-5\,\text{HH}\left[0,0,\frac{s23}{s123}\right]+5\,\text{HH}\left[1,0,\frac{s23}{s123}\right]+\text{HH}\left[0,0,0,\frac{s23}{s123}\right]-\right.$$

... and pages and pages more.

Symbol

Some of the ingredients of the answer:

- π^n, ζ_2, ζ_3, ζ_4
- $\log(z) \quad (= \int d\log(z))$
- $\mathrm{Li}_k(z) \;=\; -\int_0^z d\log(1-t) \circ \underbrace{d\log(t) \circ \cdots \circ d\log(t)}_{k-1 \text{ times}}$
- $G(a_k, a_{k-1}, \ldots; z) \;=\; \int_0^z G(a_{k-1}, \ldots; t)\, d\log(a_1 - t)$

In general - **transcendentality** ≤ 4 functions with many complicated relations between them, e.g.:

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) + \log(1-z)\log(z) \;=\; \frac{\pi^2}{6}$$

Help!

Symbol

Define a function T_k of transcendentality degree k as:

$$T_k = \int_a^b d\log R_1 \circ \cdots \circ d\log R_k$$

The **symbol**:

$$\mathcal{S}(T_k) = R_1 \otimes \cdots \otimes R_k$$

Some properties

$$R_1 \cdots \otimes (R_a R_b) \otimes \cdots R_k = R_1 \cdots \otimes (R_a) \otimes \cdots R_k + R_1 \cdots \otimes (R_b) \otimes \cdots R_k$$
$$R_1 \cdots \otimes (cR_a) \otimes \cdots R_k = R_1 \cdots \otimes (R_a) \otimes \cdots R_k$$

[Goncharov, Spradlin, Vergu, Volovich]

Symbol: example

Let's go back to Euler's identity:

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) + \log(1-z)\log(z) = \frac{\pi^2}{6}$$

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(t)\, d\log t\,, \qquad \mathrm{Li}_1(z) = -\log(1-z)$$

$$\mathcal{S}(\mathrm{Li}_2(z)) = -(1-z)\otimes z$$

$$\mathcal{S}(\log(z_1)\log(z_2)) = z_1\otimes z_2 + z_2\otimes z_1$$

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The symbol of Euler's identity:

$$-(1-z)\otimes z - z\otimes(1-z) + (1-z)\otimes z + z\otimes(1-z) = 0$$

Symbol: example

Let's go back to Euler's identity:

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) + \log(1-z)\log(z) \; = \; \frac{\pi^2}{6}$$

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The symbol of Euler's identity:

$$-(1-z)\otimes z - z\otimes(1-z) + (1-z)\otimes z + z\otimes(1-z) = 0$$

Complicated relation becomes **trivial** but we **lose some information!**
Two very **different functions** can lead to the **same symbol!**

Symbol

$$\mathcal{R}^{(2)} = \mathcal{R}^{(2)}_{\text{BPS}} + \mathcal{R}^{(2)}_{\text{non BPS}} \qquad \mathcal{R}^{(2)}_{\text{non BPS}} = \frac{(18-\pi^2)}{\epsilon} + \sum_{i=0}^{3} \mathcal{R}^{(2)}_{\text{non-BPS};3-i}$$

$$\mathcal{S}^{(2)}_{\text{BPS}}(u,v,w) = u \otimes v \otimes \left[\frac{u}{w} \otimes_S \frac{v}{w}\right] + \frac{1}{2} u \otimes \frac{u}{(1-u)^3} \otimes \frac{v}{w} \otimes \frac{v}{w}$$

$$\mathcal{S}^{(2)}_{3}(u,v,w) = -2\left[u \otimes (1-u) \otimes \frac{u}{1-u} + u \otimes u \otimes \frac{v}{1-u} + u \otimes v \otimes \frac{uv}{w^2}\right]$$

$$u = \frac{s_{12}}{q^2}\,, \qquad v = \frac{s_{23}}{q^2}\,, \qquad w = \frac{s_{31}}{q^2}$$

Name of the game

Symbol does not lead to a unique function! Construct **simpler** function with same symbol and numerically fix beyond-the-symbol terms.

Two-loop remainder

$$\mathcal{R}^{(2)}_{\text{BPS}} = \frac{3}{2}\text{Li}_4(u) - \frac{3}{4}\text{Li}_4\left(-\frac{uv}{w}\right) + \frac{3}{2}\log(w)\text{Li}_3\left(-\frac{u}{v}\right) - \frac{1}{16}\log^2(u)\log^2(v)$$
$$-\frac{\log^2(u)}{32}\left[\log^2(u) - 4\log(v)\log(w)\right] - \frac{\zeta_2}{8}\log(u)[5\log(u) - 2\log(v)]$$
$$-\frac{\zeta_3}{2}\log(u) - \frac{7}{16}\zeta_4 + \text{perms}\,(u,v,w)$$

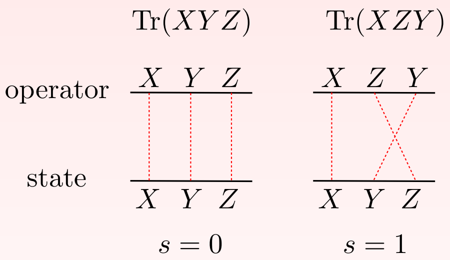
$$\mathcal{R}^{(2)}_{\text{non BPS;3}} = 2\left[\text{Li}_3(u) + \text{Li}_3(1-u)\right] - \frac{1}{2}\log^2(u)\log\frac{vw}{(1-u)^2}$$
$$+\frac{2}{3}\log(u)\log(v)\log(w) + \frac{2}{3}\zeta_3 + \text{perms}\,(u,v,w)$$

$$\mathcal{R}^{(2)}_{\text{non BPS;2}} = -12\left[\text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w)\right] - 2\log^2(uvw) + 36\zeta_2$$

$$\mathcal{R}^{(2)}_{\text{non BPS;1}} = -12\log(uvw) \qquad \mathcal{R}^{(2)}_{\text{non BPS;0}} = 126$$

Leading transcendentality contained in the BPS part!

Two-loop remainder: observations



Maximum transcendentality of a term
=
4 − # shuffles

[Loebbert et al.]

- The half-BPS $\mathrm{Tr}(X\{Y,Z\})$ contains the no-shuffle case - terms of **transcendentality 4**
- The offset $\mathrm{Tr}(XZY)$ involves one shuffle: max. **transcendentality 3**
- Computing form factors of half-BPS operators **very useful** also for other operators!

Two-loop remainder: comparison to $SU(2)$

$SU(2)$ two-loop spin chain: [Loebbert, Nandan, Sieg, Wilhelm, Yang]

- "Open" - no trace in the operator
- No length-changing interactions
- Smaller subsector: two letters $\{X, Y\}$

Surprising observation: compare the remainders

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non BPS};3} = \sum_{S_3} \left(R_i^{(2)}\right)^{XYX}_{XXY}\Big|_3 - 6\,\zeta_3 \,,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non BPS};2} = \sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_2 - 5\pi^2 \,,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non BPS};1,0} = \sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_{1,0} .$$

- **Universality?**

Two-loop remainder: comparison to $SU(2)$

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Surprising observation: compare the remainders

remainder density

$$\tfrac{1}{2}\mathcal{R}^{(2)}_{\text{non BPS};3} = \sum_{S_3} \left(R_i^{(2)}\right)^{XYX}_{XXY}\Big|_3 - 6\,\zeta_3 \,,$$

transcendentality 2 part

$$\tfrac{1}{2}\mathcal{R}^{(2)}_{\text{non BPS};2} = \sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_2 - 5\pi^2 \,,$$

$$\tfrac{1}{2}\mathcal{R}^{(2)}_{\text{non BPS};1,0} = \sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_{1,0} \,.$$

six permutations (u, v, w) site

- **Universality?**

Two-loop mixing

So what about $\mathcal{O}_F = \mathrm{Tr}(\psi\psi)$? Under renormalisation

$$\begin{pmatrix} \mathcal{O}_F^{\mathrm{ren}} \\ \mathcal{O}_B^{\mathrm{ren}} \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_F{}^F & \mathcal{Z}_F{}^B \\ \mathcal{Z}_B{}^F & \mathcal{Z}_B{}^B \end{pmatrix} \begin{pmatrix} \mathcal{O}_F^{\mathrm{bare}} \\ \mathcal{O}_B^{\mathrm{bare}} \end{pmatrix}$$

Determine the renormalisation constants by studying the UV-divergent parts of the following form factors:

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Determine the renormalisation constants by studying the UV-divergent parts of the following form factors:

$$(\mathcal{Z}^{(L)})_B{}^B \quad \leftrightarrow \quad F^{(L)}_{\mathrm{Tr}(X[Y,Z])}(1^{\phi^{12}}, 2^{\phi^{23}}, 3^{\phi^{31}}; q)\big|_{\mathrm{UV}} \qquad \textbf{done!}$$

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$$(\mathcal{Z}^{(L)})_B{}^B \quad \leftrightarrow \quad F^{(L)}_{\mathrm{Tr}(X[Y,Z])}(1^{\phi^{12}}, 2^{\phi^{23}}, 3^{\phi^{31}}; q)\big|_{\mathrm{UV}}$$ **done!**

$$(\mathcal{Z}^{(L)})_B{}^F \quad \leftrightarrow \quad F^{(L)}_{\mathrm{Tr}(X[Y,Z])}(1^{\psi^{123}}, 2^{\psi^{123}}; q)\big|_{\mathrm{UV}}$$ **very easy!**

Two-loop mixing

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done!

$$(\mathcal{Z}^{(L)})_B{}^F \quad \leftrightarrow \quad F^{(L)}_{\mathrm{Tr}(X[Y,Z])}(1^{\psi^{123}}, 2^{\psi^{123}}; q)\big|_{\mathrm{UV}}$$

very easy!

$$(\mathcal{Z}^{(L)})_F{}^B \quad \leftrightarrow \quad F^{(L)}_{\mathrm{Tr}(\psi\psi)}(1^{\phi^{12}}, 2^{\phi^{23}}, 3^{\phi^{31}}; q)\big|_{\mathrm{UV}}$$

easy!

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$$(\mathcal{Z}^{(L)})_F{}^B \quad \leftrightarrow \quad F^{(L)}_{\mathrm{Tr}(\psi\psi)}(1^{\phi^{12}}, 2^{\phi^{23}}, 3^{\phi^{31}}; q)\big|_{\mathrm{UV}} \qquad \textbf{easy!}$$

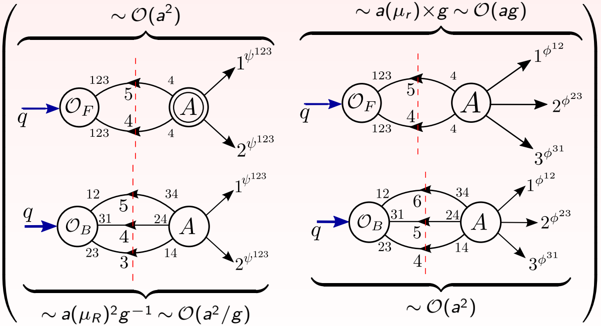
$$(\mathcal{Z}^{(L)})_F{}^F \quad \leftrightarrow \quad F^{(L)}_{\mathrm{Tr}(\psi\psi)}(1^{\psi^{123}}, 2^{\psi^{123}}; q)\big|_{\mathrm{UV}} \qquad \textbf{recycled!}$$

Two-loop dilatation operator

The quantum corrections to the dilatation operator

$$\delta\mathfrak{D} = -\mu_R \frac{\partial}{\partial \mu_R} \log \mathcal{Z} \qquad\qquad a(\mu_R) \ := \ \frac{g^2 N e^{-\epsilon\gamma}}{(4\pi)^{2-\epsilon}} \left(\frac{\mu_R}{\mu}\right)^{-2\epsilon}$$

$\log(\mathcal{Z}) \sim$



Two-loop dilatation operator

$$\log(\mathcal{Z}) = \begin{pmatrix} a^2(\mu_R)\,\frac{6}{\epsilon} & -a(\mu_R) g\,\frac{6}{\epsilon} \\ -\frac{a^2(\mu_R)}{g}\cdot\frac{6}{\epsilon} & a(\mu_R)\cdot\frac{6}{\epsilon} - a^2(\mu_R)\cdot\frac{18}{\epsilon} \end{pmatrix} + \mathcal{O}(a(\mu_R)^3)$$

$$\delta\mathfrak{D} = \lim_{\epsilon\to 0}\left[-\mu_R\frac{\partial}{\partial\mu_R}\log(\mathcal{Z})\right] = 12\times\begin{pmatrix} 2\,a^2 & -ag \\ -2\,a^2/g & a-6\,a^2 \end{pmatrix}$$

[Beisert, Eden]

Two-loop dilatation operator

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[Beisert, Eden]

Eigenvectors: $\mathcal{O}_{\mathrm{BPS}'} = \mathcal{O}_F + g\,\mathcal{O}_B \qquad \mathcal{O}_{\mathcal{K}} = \mathcal{O}_B - \frac{gN}{8\pi^2}\mathcal{O}_F$

Eigenvalues: $\gamma_{\mathrm{BPS}'} = 0 \qquad \gamma_{\mathcal{K}} = 12\,a - 48\,a^2 + \mathcal{O}(a^3)$

Conclusions and outlook

- **Takeaway message**:
    - Important motivation for studying form factors of half-BPS operators
    - Connection between QCD Higgs amplitudes and $\mathcal{N} = 4$ form factors
    - Unexpected matching of form factors of operators in different sectors
    - Equally interesting connection between form factors and dilatation operator - possible to compute $\delta\mathfrak{D}$ using on-shell methods
- **Future directions**:
    - Maximally transcendental part of $\mathrm{Tr}(F^3)$ given by half-BPS $\mathrm{Tr}(\phi^3)$?
    - Investigate multiplet structure for $\mathrm{Tr}(F^3)$ (c.f. $\mathrm{Tr}(F^2)$ vs. $\mathrm{Tr}(\phi^2)$)
    - Develop a more systematic connection between Higgs-gluon amplitudes and form factors in $\mathcal{N} = 4$ SYM
    - Investigate higher-dimensional operators in $\mathcal{L}_{\mathrm{eff}}$
    - Explore the theme of universality across sectors of $\mathcal{N} = 4$
    - Compute $\delta\mathfrak{D}$ for the whole theory or at higher loop order

Conclusions and outlook

Thank you!

Extras: spinor-helicity formalism

$$p^{\alpha\dot\alpha} = \sigma_\mu^{\alpha\dot\alpha} p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}, \qquad \sigma_\mu^{\alpha\dot\alpha} = (1, \sigma_i)$$

then $det(p) = p_\mu p^\mu = m^2$

For massless particles: $det(p^{\alpha\dot\alpha}) = 0$ hence $p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} \qquad (h = -/+)$

Raise/lower indices: $\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta$

Create invariants: $\langle\lambda_i\lambda_j\rangle = \lambda_i^\alpha \lambda_{j\,\alpha} = \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta =: \langle ij\rangle\,, \quad \tilde\lambda_i^{\dot\alpha}\tilde\lambda_{j\,\dot\alpha} =: [ij]$

and **Mandelstam invariants**: $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = \langle ij\rangle\,[ji]$

E.g.: first non-vanishing amplitude **(MHV)**: [Parke, Taylor]

$$\mathcal{A}(1^+, \ldots, i^-, j^-, \ldots, n^+) = i\,\frac{\langle ij\rangle^4}{\langle 12\rangle\,\langle 23\rangle \ldots \langle n1\rangle}$$