Selected Topics from Analytical Foundations of Quasiconformal Mappings

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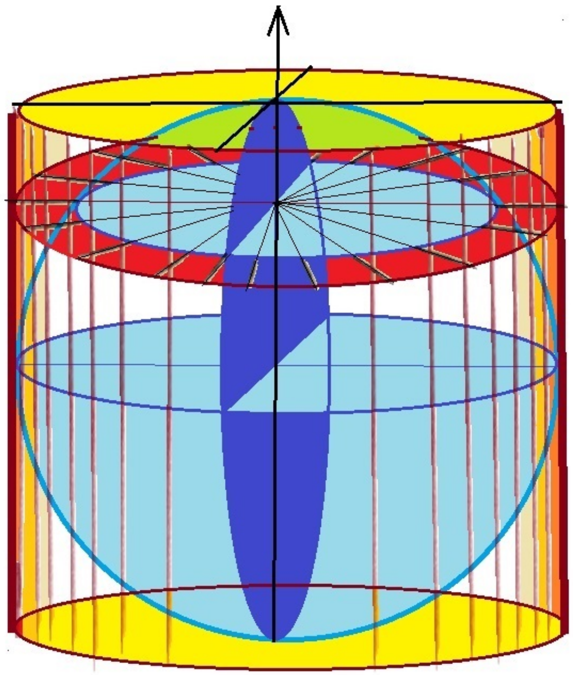
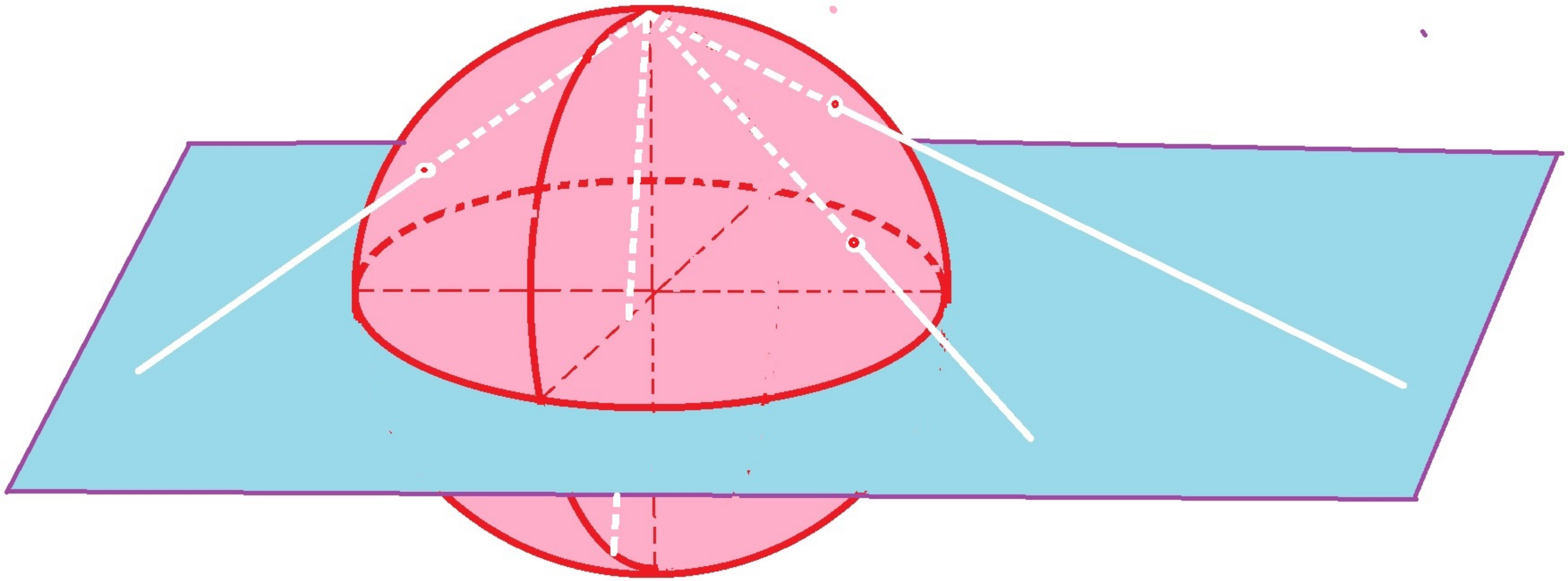
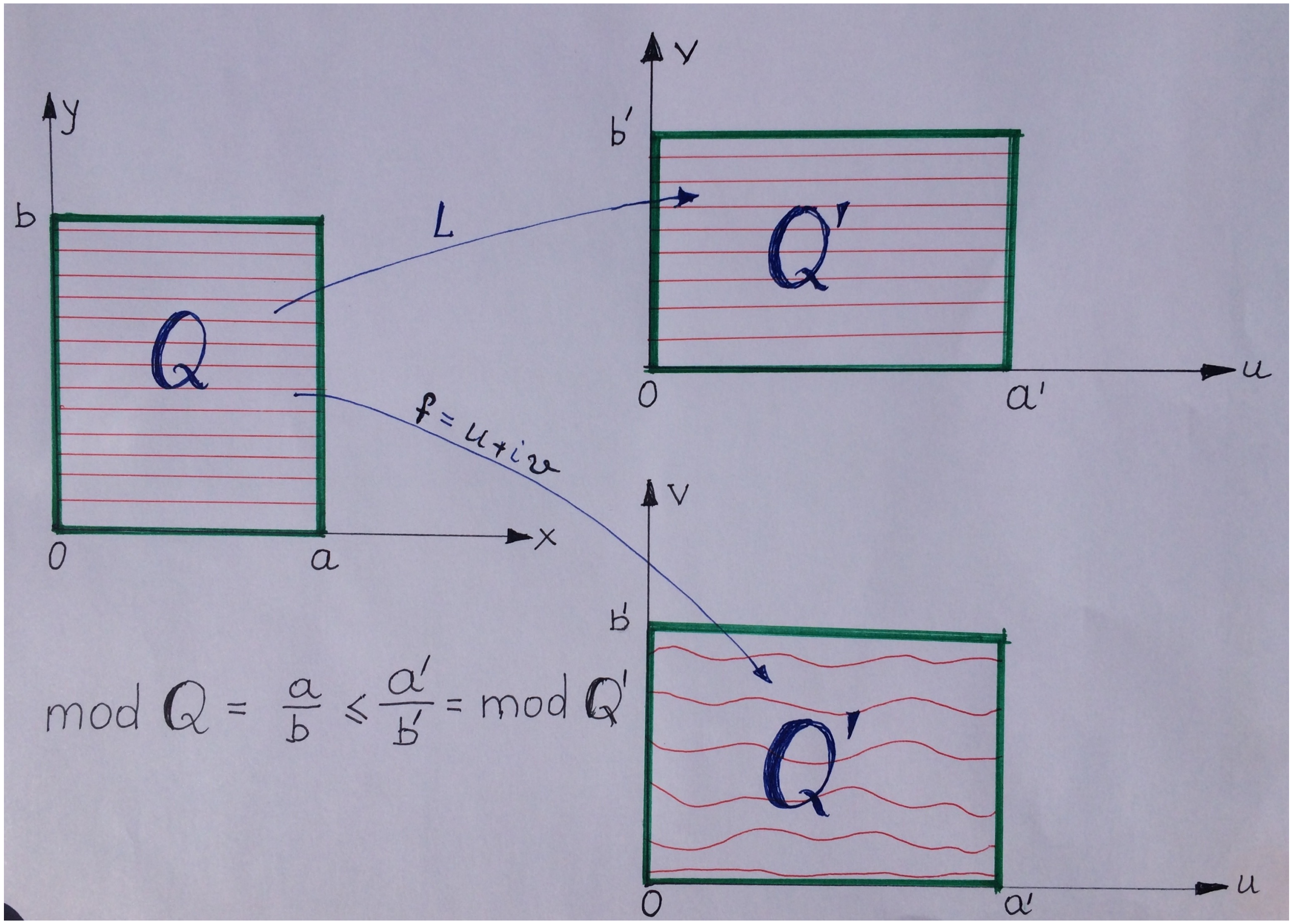


The Mercator projection is a cylindrical map projection presented by the Flemish (Dutch-Belgium community) geographer and cartographer Gerardus Mercator in 1569 as a navigation tool.



H. Grötzsch (1928) Problem



Let $f = u + iv \colon Q \xrightarrow{\text{onto}} Q'$ be homeomorphism in $\mathscr{W}^{1,1}(Q, Q')$ (or in $\mathscr{W}^{1,2}(Q, Q')$). Distortion:

$$K_f = \frac{|Df|^2}{J_f} \qquad \text{(operator norm)}$$

Theorem. *The linear map*

$$L(x, y) = \frac{a'}{a} x + i \frac{b'}{b} y$$

has smallest $\mathscr{L}^1$-mean distortion.

Proof. By the method of free-Lagrangians

1) Free Lagrangians

$$\iint_Q J_f \leqslant |f(Q)| = |Q'| = a'b'$$

$$\begin{aligned}\iint_Q (\operatorname{Re} f_x) &= \operatorname{Re} \int_0^b \left(\int_0^a f_x(x,y)\,dx \right) dy \\ &= \int_0^b \operatorname{Re}[f(a,y) - f(0,y)]\,dy = a' \cdot b\end{aligned}$$

Similarly,

$$\iint_Q \operatorname{Im} f_y = b' \cdot a\,.$$

2)

$$|DL| = \max\{\frac{a'}{a}, \frac{b'}{b}\}\,.$$

We may, and do, assume that

$$|DL| = \frac{a'}{a} = \operatorname{Re} L_x \quad \text{(or } \operatorname{Im} L_y\text{)}$$

Hence

$$\operatorname{Re} \iint_Q (f_x - L_x) = 0$$

$$\iint_Q (J_f - J_L) \leqslant 0$$

3) Sharp free-Lagrangian inequality

$$K_f - K_L = \frac{|Df|^2}{J_f} - \frac{|DL|^2}{J_L} \geqslant \qquad \text{by polyconvexity}$$

$$+ \frac{2|DL|}{J_L}(|Df| - |DL|)$$

$$- \frac{|DL|^2}{J_L^2}(J_f - J_L)$$

or by $(|Df|J_L - |DL|J_f)^2 \geqslant 0$

$$\geqslant C_1 \operatorname{Re}(f_x - L_x) - C_2(J_f - J_L)$$

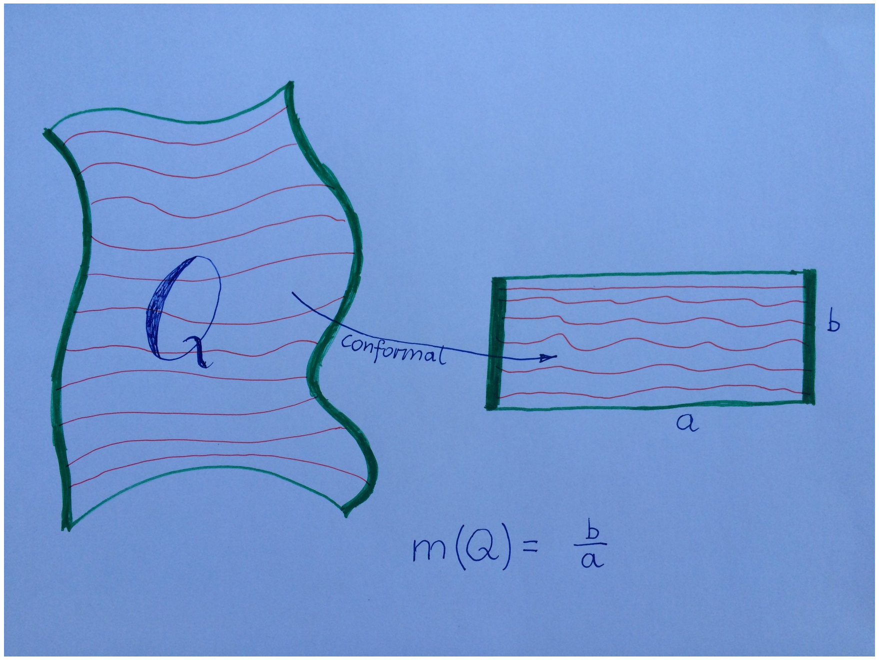
Hence

$$\iint_Q (K_f - K_L) \geqslant 0$$

as desired.

Quadrilaterals

-*Jordan domains* Q *with a pair of disjoint arcs on* ∂Q

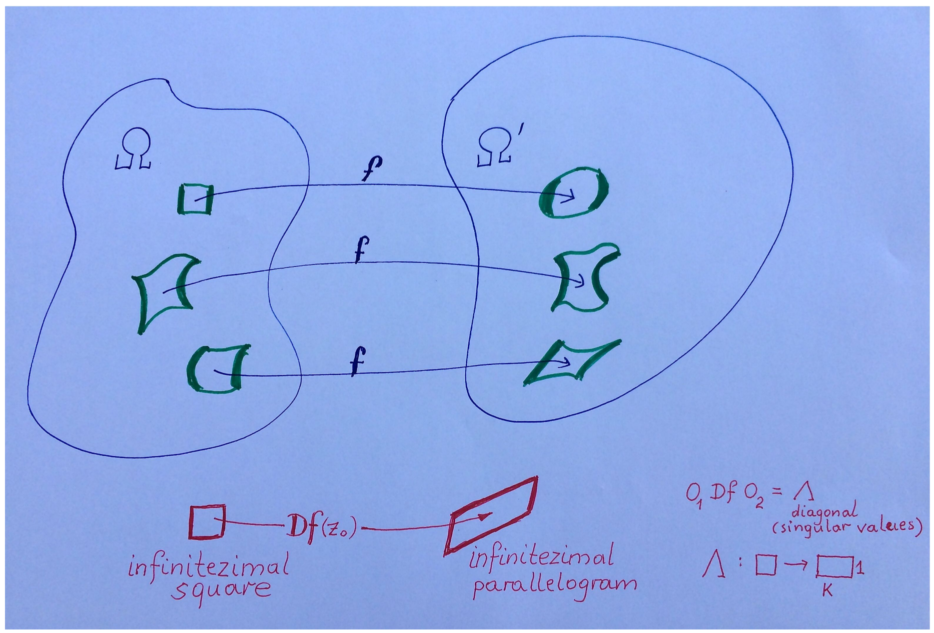


$$m(Q) = \frac{a}{b}\,.$$

Definition (Grötzsh or geometric approach)

A mapping $f\colon \Omega \to \Omega'$ *is* K*-quasiconformal,* $1 \leqslant K < \infty$*, if*

$$\frac{1}{K} m(Q) \leqslant m[f(Q)] \leqslant K m(Q)$$



Analytic Definition

Every homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$ *is differentiable a.e. (D. Menchoff, 1931) and (F.W. Gehring - O.Lehto, 1959). Looking at the infitniitzimal quadrilaterals at the points of differentiability we find that the geometric definition implies*

$$|Df|^2 \leqslant K J_f$$

The ratio of singular values is $\leqslant K$.

The Riemann Mapping Theorem

Conformal type of a domain of connectivity $\ell > 2$ is determined by $3\ell - 6$ parameters (moduli of the domain); that is, two ℓ-connected domains are conformally equivalent if and only if they agree in all $3\ell - 6$ moduli. As for the bounded doubly connected domains, we have the Schottky Theorem (1877): A conformal mapping

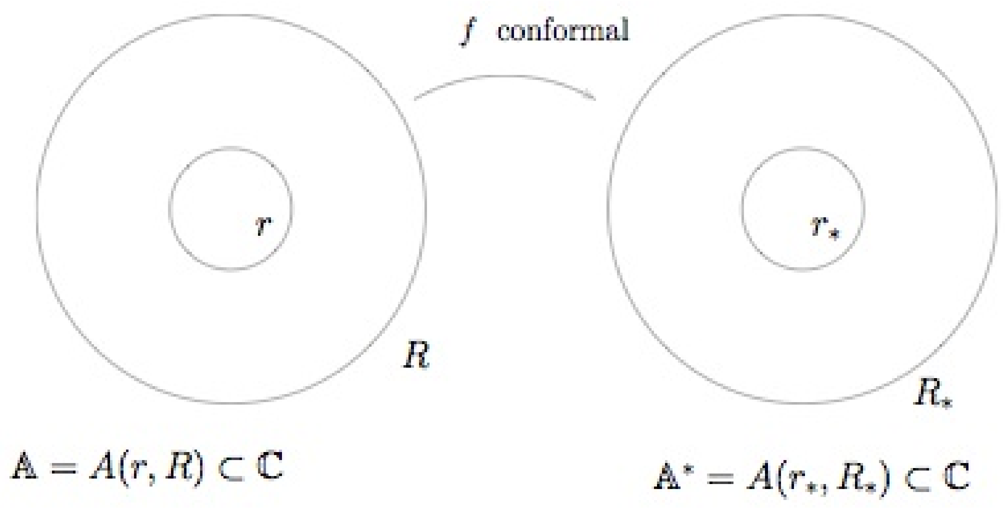
$$\mathsf{h} : \mathbb{A} = A(r, R) \quad \xrightarrow{\text{onto}} \quad A(r_*, R_*) = \mathbb{A}^*$$

between circular annuli exists if and only if

$$\text{Mod}\, \mathbb{A}^* := \log \frac{R_*}{r_*} = \log \frac{R}{r} := \text{Mod}\, \mathbb{A}$$

Harmonic mappings of doubly connected domains in the complex plane, being next in the order of complexity (after simply connected case) are of great interest.

Schottky's Theorem via Free-Lagrangians



A conf. $f\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ exists iff $m\mathbb{A} = \log \frac{R}{r} = \log \frac{R_*}{r_*} = m\mathbb{A}^*$.

$$\left(f(z) = \frac{r_*}{r} z \quad \text{or} \quad f(z) = \frac{r_* R}{z} \right)$$

Proof. The Cauchy-Riemann system $f_{\bar z} = 0$ in polar coordinates read as

$$\frac{1}{\rho}\frac{\partial f}{\partial \theta} \;=\; i\,\frac{\partial f}{\partial \rho}, \qquad\qquad \text{for} \quad z = \rho\, e^{i\theta}.$$

We denote the LHS by f_T (tangential) and $\partial f/\partial\rho$ by f_N (normal). Then

$$J(z,f) = \operatorname{Im}(f_T\overline{f_N}) = |f_N|^2 = |f_T|^2. \tag{1}$$

Claim. If a homeomorphism $f\colon \mathbb{A} = A(r,R) \xrightarrow{\text{onto}} A(r_*,R_*) = \mathbb{A}^*$ belongs to $\mathscr{W}^{1,2}(\mathbb{A},\mathbb{A}^*)$ and satisfies (1), then $R/r = R_*/r_*$.

$$\frac{\sqrt{J(z,f)}}{|z|\;|f(z)|} = \left\{\begin{array}{c} \left|\frac{f_N}{\rho\, f}\right| \\ \\ \left|\frac{f_T}{\rho\, f}\right| \end{array}\right. \geqslant \left\{\begin{array}{l} \operatorname{Re}\frac{f_N}{\rho\, h} \\ \\ \operatorname{Im}\frac{f_T}{\rho\, f} \end{array}\right.$$

After integrating

$$\left(\int_{\mathbb{A}} \frac{\sqrt{J(z,h)}\,\mathrm{d}z}{|z|\,|f(z)|}\right)^2 \geqslant \begin{cases} \left(\int_{\mathbb{A}} \frac{|f|_N}{\rho\,|f|}\right)^2 \\ \\ \left(\int_{\mathbb{A}} \operatorname{Im} \frac{f_T}{\rho\, f}\right)^2 \end{cases} = \begin{cases} \left(\pm 2\pi \log \frac{R_*}{r_*}\right)^2 \\ \\ \left(\pm 2\pi \log \frac{R}{r}\right)^2 . \end{cases}$$

On the other hand,

$$\left(\int_{\mathbb{A}} \frac{\sqrt{J(z,h)}\,\mathrm{d}z}{|z|\,|f(z)|}\right)^2 \leqslant \int_{\mathbb{A}} \frac{\mathsf{d}z}{|z|^2} \cdot \int_{\mathbb{A}} \frac{J(z,f)}{|f(z)|^2}\mathsf{d}z = 2\pi \log \frac{R}{r} \cdot 2\pi \log \frac{R_*}{r_*}.$$

Therefore, the claim follows.

Another proof of Schottky's Theorem

Let $h\colon A(1,R) \xrightarrow{\text{onto}} A(1,R_*)$ be conformal. Consider

$$U(\rho) = \fint_{\mathbb{T}_\rho} |h|^2 \quad 1 \leqslant \rho < R.$$

Then the second order differential operator

$$\mathcal{L}[U] := \frac{1}{\rho}\frac{d}{d\rho}\left[\rho^3 \frac{d}{d\rho}\left(\frac{U}{\rho^2}\right)\right] \geqslant 0.$$

Therefore,

$$\rho^3 \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{U}{\rho^2}\right) \geqslant \rho^3 \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{U}{\rho^2}\right)\Big|_{\rho=1} = \dot{U}(1) - 2\,U(1) = \dot{U}(1) - 2\,.$$

Since ($h_\rho = \frac{i}{\rho} h_\theta$ Cauchy-Riemann)

$$\dot{U}(1) \; = 2 \;\mathrm{Re} \fint_{\mathbb{T}_1} \bar{h}\, h_\rho = 2 \;\mathrm{Im} \fint_{\mathbb{T}_1} \bar{h}\, h_\theta = 2 \;\mathrm{Im} \fint_{\mathbb{T}_1} \frac{h_\theta}{h} = 2\,.$$

the function $\rho \to \rho^{-2} U(\rho)$ is nondecreasing and hence $U(\rho) \geqslant \rho^2$.

Complex notation

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$Df(z)h = f_z h + f_{\bar{z}}\bar{h}$$

$$\sup_h |Df(z)h| = |f_z| + |f_{\bar{z}}| \qquad \inf_h |Df(z)h| = |f_z| - |f_{\bar{z}}|$$

$$J(z,h) = |f_z|^2 - |f_{\bar{z}}|^2$$

The basic Beltrami equation

The distortion inequality

$$|Df(z)|^2 \leqslant K J(z,f) \qquad 1 \leqslant K < \infty \,.$$

reads as

$$(|f_z| + |f_{\bar z}|)^2 \leqslant K \left(|f_z|^2 - |f_{\bar z}|^2\right)$$

$$\frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \leqslant K$$

Let

$$\mu(z) = \begin{cases} \frac{f_{\bar{z}}}{f_z} & \text{if } f_z \neq 0 \\ 0 & \text{if } f_z = 0 \end{cases}$$

Then

$$\frac{1+|\mu(z)|}{1-|\mu(z)|} \leqslant K$$

$$|\mu(z)| \leqslant \frac{K-1}{K+1} = k\,, \quad 0 \leqslant k < 1\,.$$

$$f_{\bar{z}} = \mu(z)\, f_z \qquad\qquad z \in \Omega$$

We look for all solutions $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$.