Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

Thursday, 15th January 2015, 9:30 a.m.- 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Mathematical Methods

Question 1

Let

$$Ly(x) \equiv (x+1)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0. \tag{1}$$

(a) Find the general solution of the inhomogeneous equation

$$Ly(x) = 0. \tag{2}$$

with Ly as in (1).
Hint: Show that $y_1(x) = (x+1)^2$ *is a solution, then seek a second independent solution of the form* $y_2(x) = u(x)y_1(x)$.

[8 marks]

(b) Determine the Green's function for the initial value problem

$$\begin{aligned} &Ly(x) = f(x) \\ &y(0) = \alpha, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(0) = \beta. \end{aligned}$$

[9 marks]

(c) Use the Green's function obtained in (b) to write down the solution for general data $\{f(x), \alpha, \beta\}$, where $f(x)$ is a given function, and α and β are given real constants.

[8 marks]

Question 2

(a) Consider the operator

$$Lu(x) \equiv \alpha(x)u'''(x) + u(x). \tag{3}$$

where $\alpha \in C^\infty(\mathbb{R})$ is a given smooth, strictly positive function. Here $'$ denotes differentiation with respect to x.

(i) On the interval $a < x < b$ with boundary conditions

$$u(a) = 0, \quad u(b) = 0, \quad u'(a) = 0,$$

find the operator and boundary conditions for the adjoint problem.

[6 marks]

(ii) For u a *distribution*, show that Lu is a distribution, by stating and verifying the necessary conditions.

[8 marks]

(iii) What does it mean for u to be a:

(α) distributed solution of $Lu = f$, for f a given distribution?

(β) classical solution of $Lu = f$, for f a given smooth function?

[3 marks]

(b) Prove that

$$\alpha(x)\delta'(x) = -\alpha'(0)\delta(x) + \alpha(0)\delta'(x)$$

in the distributional sense, where $\alpha \in C^\infty(\mathbb{R})$ and δ denotes the delta distribution.

[8 marks]

Question 3

(a) Define the operator

$$Ly(x) \equiv 3\int_0^1 t^2 y(t)\ dt + 3\int_0^1 x^2 y(t)\ dt.$$

By taking the appropriate inner product show that L is self-adjoint. **[6 marks]**

(b) Show that the integral equation

$$y(x) = \lambda \int_0^1 xt(x+t)y(t)\ dt$$

only has a non-trivial solution if λ is a root of

$$\lambda^2 + 120\lambda - 240 = 0.$$

[9 marks]

(c) Consider the integral equation

$$y(x) = \mu \int_0^\pi \sin(x+t)y(t)\ dt + f(x).$$

For which values of the constant μ does there exist a unique solution for arbitrary $f(x)$?

[10 marks]

Question 4

We consider here the hypergeometric equation

$$x(1-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + [c-(a+b+1)x]\,\frac{\mathrm{d}y}{\mathrm{d}x} - aby(x) = 0, \tag{4}$$

where a, b, c are given real numbers.

(a) Find and classify any singular points. **[7 marks]**

(b) Consider a series solution

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}. \tag{5}$$

to (4) about $x = 0$. Find a condition on c under which we can be certain to have *two linearly independent* solutions of the form (5).

[8 marks]

(c) Let $c = \frac{3}{2}$. Give all conditions on a, b under which:

(i) One solution is a polynomial. **[10 marks]**

(ii) One solution is equal to

$$y(x) = \frac{1}{1-x}$$

for $|x| < 1$.

You may use without proof the power series expansion

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,$$

valid for $|x| < 1$.

Section B — Further Mathematical Methods

Question 5

Consider the functional

$$J[y] = \int_0^L F(x, y(x), y'(x))dx, \quad F(x, y, y') = \frac{1}{2}\left(y'\right)^2 - \frac{1}{2}y^2 + \frac{1}{4}y^4 - \alpha y, \tag{6}$$

where $L > 0$ and α are constants. (Note that $' = d/dx$.)

(a) Derive the Euler-Lagrange equation and the appropriate boundary conditions for the extremals y of J, stating the Fundamental Lemma of Calculus of Variations without proof. **[7 marks]**

(b) Determine the corresponding Hamilton function $H = py' - F$ in terms of the variables $q = y$, and $p = \partial F/\partial y'$. For this H, show that Hamilton's equations $p' = -\partial H/\partial q$, $q' = \partial H/\partial p$ have the form

$$\frac{dp}{dx} = g(q) - \alpha, \quad \frac{dq}{dx} = p, \tag{7}$$

and determine $g(q)$ explicitly.

Show that if y solves the Euler-Lagrange equation for (6), then p and q satisfy Hamilton's equations (7), and moreover $dH/dx = 0$. **[7 marks]**

(c) Show that for $-2\sqrt{3}/9 < \alpha < 2\sqrt{3}/9$, the system (7) has exactly three critical points $(q, p) = (q_-, 0)$, $(q_0, 0)$, $(q_+, 0)$, with $q_- < q_0 < q_+$, and classify them by inspecting the linearised system at each of the three critical points.

Are paths of the nonlinear system (7) close to the middle critical point $(p, q) = (q_0, 0)$ periodic? Give a brief reason for your answer.

[Hint: The bounds for α can be found without determining q_-, q_0 and q_+ explicitly, by using properties of g. A sketch of g may be helpful.] **[5 marks]**

(d) Give a qualitative sketch of the (q, p) phase plane for the case where $0 < \alpha < 2\sqrt{3}/9$, including the location of the critical points relative to $q = \pm 1$ and $q = 0$, and all paths that meet at the saddle points. Use arrows to indicate the direction of flow along all paths in the sketch. **[6 marks]**

Question 6

Consider the system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -y\left(x^2 + y^2 - 2\varepsilon\right) - x + x^3, \tag{8}$$

where $-1 < \varepsilon < 1$ is a (real) parameter.

(a) Find all the critical points of (8). **[3 marks]**

(b) By inspecting the linearised system, classify all the critical points of (8) depending on the parameter ε. What type of bifurcation occurs at the origin when $\varepsilon = 0$? **[8 marks]**

(c) Use the Poincaré-Lindstedt method to find the periodic orbit when $0 < \varepsilon \ll 1$. Determine the frequency up to, and including terms of $O(\varepsilon)$, and the corresponding value of the amplitude.

[*You may use one or more of the identities*

$$\sin^3\tau = \frac{3\sin\tau}{4} - \frac{\sin 3\tau}{4}, \qquad \sin^2\tau\cos\tau = \frac{\cos\tau}{4} - \frac{\cos 3\tau}{4},$$

$$\cos^2\tau\sin\tau = \frac{\sin\tau}{4} + \frac{\sin 3\tau}{4}, \qquad \cos^3\tau = \frac{3\cos\tau}{4} + \frac{\cos 3\tau}{4},$$

without proof.] **[14 marks]**