Part B Mathematics and Philosophy 2023-24

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Contents

1 Foreword

This Supplement to the Course Handbook specifies the Mathematics courses available for Part B in Mathematics & Philosophy in the 2024 examination. It should be read in conjunction with the Handbook for Mathematics & Philosophy for the academic year 2023-2024, to be issued in Michaelmas Term. The Handbook contains in particular information on the format and rubrics for written examination papers in Mathematics, and the classification rules applicable to Part B.

See the current edition of the Examination Regulations (`https://examregs.admin.ox.ac.uk/`) for the full regulations governing the examinations.

Part B of the Honour School of Mathematics & Philosophy

The following is reproduced from the *Examination Regulations* applicable to the 2024 examinations.

The examination for Part B shall consist of units in Mathematics and subjects in Philosophy. The schedule of units in *Mathematics* shall be published on the Mathematical Institute website by the beginning of the Michaelmas Full Term in the academic year of the examination concerned, after consultation with the Mathematics Teaching Committee. The schedule shall be in two parts: Schedule 1 (standard units) and Schedule 2 (additional units). In *Philosophy* the subjects shall be subjects from the list given in *Special Regulations for All Honour Schools Including Philosophy.*

Each candidate shall offer:

(i) Four units of *Mathematics* from Schedule 1, two of which shall be B1.1 *Logic* and B1.2 *Set Theory*,

(ii) three subjects in *Philosophy* from 101-116, 120, 122, 124, 125, 127-129, 137-139 and 198 of which two must be 122 and **either** 101 **or** 102, and

(iii) **either** two further units in *Mathematics* drawn from Schedules 1 and 2 combined **or** one further subject in *Philosophy* from subjects 101-116, 120, 122, 124, 125, 127-129, 137-139, 198 and 199: *Thesis.*

Note that the units listed under Schedule 2 are not available to those who wish to offer a total of four Philosophy subjects.

The Schedules of Mathematics units for Mathematics & Philosophy

All units in Mathematics are drawn from the list of options for Mathematics Part B.

Schedule 1 comprises those Mathematics Department courses for which the core and options in Mathematics & Philosophy Part A provide the requisite background.

Schedule 2 contains an Extended Essay option and certain further courses from Mathematics Part B appropriate for the Joint School.

In addition you may apply for special approval to be examined in Mathematics Department units not included under Schedule 1; any such subject approved will be treated as falling under Schedule 2. For the procedure for seeking approval, see below.

For the 2024 examination, the Schedules are as follows. (N.B. All topics listed are units unless otherwise stated).

Schedule 1

- B1.1 Logic [Compulsory] (MT)
- B1.2 Set Theory [Compulsory] (HT)
- B2.1 Introduction to Representation Theory (MT)
- B2.2 Commutative Algebra (HT)
- B2.3 Lie Algebras (MT)
- B3.1 Galois Theory (MT)
- B3.2 Geometry of Surfaces (MT)
- B3.3 Algebraic Curves (HT)
- B3.4 Algebraic Number Theory (HT)
- B3.5 Topology and Groups (MT)
- B4.1 Functional Analysis I (MT)
- B4.2 Functional Analysis II (HT)
- B4.3 Distribution Theory (MT)
- B4.4 Fourier Analysis (HT)
- B8.1 Probability, Measure and Martingales (MT)
- B8.2 Continuous Martingales and Stochastic Calculus (HT)
- B8.4 Information Theory (MT)
- B8.5 Graph Theory (MT)
- SB3.1 Applied Probability (HT)

Schedule 2 (additional units)

- BEE "Mathematical" Extended Essay [double unit] (MT/HT)
- BO1.1 History of Mathematics [double unit] (MT/HT)
- BOE "Other Mathematical" Extended Essay [double unit] (MT/HT)
- Lambda Calculus and Types (HT)
- Computational Complexity (HT)
- Combinatorial Optimisation (MT)

Any other unit course from the list of Mathematics Department units in Part B for which special approval has been granted.

Procedure for seeking approval of additional options where this is required

You may, if you have the support of your Mathematics tutor, apply to the Chairman of the Joint Committee for Mathematics and Philosophy for approval of one or more other options from the list of Mathematics Department units for Part B, including the Statistics and Computer Science options. This list can be found in the schedule of units for Mathematics Part B.

Applications for special approval must be made through the candidate's college and emailed to the Chair of the Joint Committee for Mathematics and Philosophy, c/o Haleigh Bellamy, Mathematical Institute, to arrive by **Friday of Week 5 of Michaelmas Term**. Be sure to consult your college tutors if you are considering asking for approval to offer one of these additional options.

Given that each of these additional options, which are all in applied mathematics, presume facility with some or other results and techniques covered in first, second or third year Mathematics courses not taken by Mathematics & Philosophy candidates, such applications will be exceptional.

You should also be aware that there may be a clash of lectures for specially approved options and those listed in Schedules 1 and 2 and with lectures in Philosophy.

Registration for Part B Mathematics courses 2023-24

Students will be asked to register for the options they intend to take by the end of week 11, Trinity Term 2023. It is helpful if their registration is as accurate as possible as the data is used to make teaching resource arrangements. Towards the start of the academic year students will be given the opportunity to make edits to their course registration. Students will then be asked to sign up for classes via the Teaching Management System (`https://tms.ox.ac.uk/`) at the start of Michaelmas Term 2023. Further information about this will be sent via email before the start of term.

Students who register for a course or courses for which there is a quota should consider registering for an additional course (by way of a "reserve choice") in case they do not receive a place on the course with the quota. They may also have to give the reasons why they wish to take a course which has a quota, and provide the name of a tutor who can provide a supporting statement for them should the quota be exceeded. Where this is necessary students will be contacted by email after they have registered. In the event that the quota for a course is exceeded, the Mathematics Teaching Committee will decide who may have a place on the course on the basis of the supporting statements from the student and tutor, and all relevant students will be notified of the decision by email.

2 B1.1 Logic

2.1 General Prerequisites

There are no formal prerequisites, but familiarity with basic mathematical objects and notions will be helpful at some points in the course, in particular with some examples. In particular, such objects/notions as: the rational, real and complex number fields; sets; the idea of surjective, injective and bijective functions; order relations; the definitions of basic abstract mathematical structures such as groups and fields (all covered in Mathematics I and II in Prelims).

2.2 Overview

To give a rigorous mathematical treatment of the fundamental ideas and results of logic that is suitable for the non-specialist mathematicians and will provide a sound basis for more advanced study. Cohesion is achieved by focusing on the Completeness Theorems and the relationship between provability and truth. Consideration of some implications of the Compactness Theorem gives a flavour of the further development of model theory. To give a concrete deductive system for predicate calculus and prove the Completeness Theorem, including easy applications in basic model theory.

2.3 Learning Outcomes

Students will be able to use the formal language of propositional and predicate calculus and be familiar with their deductive systems and related theorems. For example, they will know and be able to use the soundness, completeness and compactness theorems for deductive systems for predicate calculus.

2.4 Synopsis

The notation, meaning and use of propositional and predicate calculus. The formal language of propositional calculus: truth functions; conjunctive and disjunctive normal form; tautologies and logical consequence. The formal language of predicate calculus: satisfaction, truth, validity, logical consequence.

Deductive system for propositional calculus: proofs and theorems, proofs from hypotheses, the Deduction Theorem; Soundness Theorem. Maximal consistent sets of formulae; completeness; constructive proof of completeness.

Statement of Soundness and Completeness Theorems for a deductive system for predicate calculus; derivation of the Compactness Theorem; simple applications of the Compactness Theorem.

A deductive system for predicate calculus; proofs and theorems; prenex form. Proof of Completeness Theorem. Existence of countable models, the downward Löwenheim-Skolem Theorem.

2.5 Reading List

1. R. Cori and D. Lascar, *Mathematical Logic: A Course with Exercises (Part I)* (Oxford University Press, 2001), sections 1, 3, 4.
2. A. G. Hamilton, *Logic for Mathematicians* (2nd edition, Cambridge University Press, 1988), pp.1-69, pp.73-76 (for statement of Completeness (Adequacy) Theorem), pp.99-103 (for the Compactness Theorem).
3. W. B. Enderton, *A Mathematical Introduction to Logic* (Academic Press, 1972), pp.101-144.
4. D. Goldrei, *Propositional and Predicate Calculus: A model of argument* (Springer, 2005).
5. A. Prestel and C. N. Delzell, *Mathematical Logic and Model Theory* (Springer, 2010).

2.6 Further Reading

1. R. Cori and D. Lascar, *Mathematical Logic: A Course with Exercises (Part II)* (Oxford University Press, 2001), section 8.

3 B1.2 Set Theory

3.1 General Prerequisites

There are no formal prerequisites, but familiarity with some basic mathematical objects and notions such as: the rational and real number fields; the idea of surjective, injective and bijective functions, inverse functions, order relations; the notion of a continuous function of a real variable, sequences, series, and convergence, and the definitions of basic abstract structures such as fields, vector spaces, and groups (all covered in Mathematics I and II in Prelims) will be helpful at points.

3.2 Overview

Introduce sets and their properties as a unified way of treating mathematical structures. Emphasise the difference between an intuitive collection and a formal set. Define (infinite) cardinal and ordinal numbers and investigate their properties. Frame the Axiom of Choice and its equivalent forms and study their implications.

3.3 Learning Outcomes

Students will have a sound knowledge of set theoretic language and be able to use it to codify mathematical objects. They will have an appreciation of the notion of infinity and arithmetic of the cardinals and ordinals. They will have developed a deep understanding of the Axiom of Choice, Zorn's Lemma and the Well-Ordering Principle.

3.4 Synopsis

What is a set? Introduction to the basic axioms of set theory. Ordered pairs, cartesian products, relations and functions. Axiom of Infinity and the construction of the natural numbers; induction and the Recursion Theorem.

Cardinality; the notions of finite and countable and uncountable sets; Cantor's Theorem on power sets. The Tarski Fixed Point Theorem. The Schröder-Bernstein Theorem. Basic cardinal arithmetic.

Well-orders. Comparability of well-orders. Ordinal numbers. Transfinite induction; transfinite recursion [informal treatment only]. Ordinal arithmetic.

The Axiom of Choice, Zorn's Lemma, the Well-ordering Principle; comparability of cardinals. Equivalence of WO, CC, AC and ZL. Cardinal numbers.

3.5 Reading List

1. D. Goldrei, *Classic Set Theory* (Chapman and Hall, 1996).
2. H. B. Enderton, *Elements of Set Theory* (Academic Press, 1978).

3.6 Further Reading

1. R. Cori and D. Lascar, *Mathematical Logic: A Course with Exercises (Part II)* (Oxford University Press, 2001), section 7.1-7.5.
2. R. Rucker, *Infinity and the Mind: The Science and Philosophy of the Infinite* (Princeton University Press, 1995). An accessible introduction to set theory.
3. J. W. Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton University Press, 1990). For some background, you may find JW Dauben's biography of Cantor interesting.
4. M. D. Potter, *Set Theory and its Philosophy: A Critical Introduction* (Oxford University Press, 2004). An interestingly different way of establishing Set Theory, together with some discussion of the history and philosophy of the subject.
5. W. Sierpinski, *Cardinal and Ordinal Numbers* (Polish Scientific Publishers, 1965). More about the arithmetic of transfinite numbers.
6. J. Stillwell, *Roads to Infinity* (CRC Press, 2010).

4 B2.1 Introduction to Representation Theory

4.1 General Prerequisites

Rings and Modules is essential. Group Theory is recommended.

4.2 Overview

This course gives an introduction to the representation theory of finite groups. Representation theory is a fundamental tool for studying symmetry by means of linear algebra: it is studied in a way in which a given group may act on vector spaces, giving rise to the notion of a representation.

The first part of the course will deal with the structure theory of semisimple algebras and their modules (representations). We will prove that any finite-dimensional semisimple algebra is isomorphic to a product of matrix rings (Wedderburn's Theorem).

In later parts of the course we apply the developed material to group algebras, and classify when group algebras are semisimple (Maschke's Theorem). All of this material will be applied to the study of characters and representations of finite groups.

4.3 Learning Outcomes

They will know in particular simple modules and semisimple algebras and they will be familiar with examples. They will appreciate important results in the course such as Schur's Lemma, Wedderburn's Theorem, the Row Orthogonality Theorem and Burnside's $p^\alpha \ q^\beta$ Theorem. They will be familiar with the classification of semisimple algebras over $\mathbb{C}$ and be able to apply this to representations and characters of finite groups.

4.4 Synopsis

Representations of groups. Maschke's Theorem. The group ring.

Modules and their relationship with representations. Semisimple algebras, Schur's Lemma and the Wedderburn Theorem. Characters of complex representations. Orthogonality relations, finding character tables. Tensor product of representations. Induction and restriction of representations. Application: Burnside's $p^\alpha \ q^\beta$ Theorem.

4.5 Reading List

1. G. D. James and M. Liebeck, *Representations and Characters of Finite Groups* (2nd edition, Cambridge University Press, 2001).

2. J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics 42 (Springer-Verlag, 1977).

4.6 Further Reading

1. J. L. Alperin and R. B. Bell, *Groups and Representations*, Graduate Texts in Mathematics 162 (Springer-Verlag, 1995).

2. P. M. Cohn, *Classic Algebra* (Wiley & Sons, 2000). (Several books by this author available.)

3. C. W. Curtis, and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras* (Wiley & Sons, 1962).

4. L. Dornhoff, *Group Representation Theory* (Marcel Dekker Inc., New York, 1972).

5. I. M. Isaacs, *Character Theory of Finite Groups* (AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island, 2006).

6. P. Etingof, *Introduction to representation theory* (Online course notes, MIT 2011).

7. K. Erdmann, T. Holm *Algebras and Representation Theory*, Springer Undergraduate Mathematical Series (2018), ISSN 1615-2085

5 B2.2 Commutative Algebra

5.1 General Prerequisites

Rings and Modules is essential. Galois Theory is strongly recommended.

5.2 Overview

Amongst the most familiar objects in mathematics are the ring of integers and the polynomial rings over fields. These play a fundamental role in number theory and in algebraic geometry, respectively. The course explores the basic properties of such rings.

5.3 Synopsis

Modules, ideals, prime ideals, maximal ideals. Noetherian rings; Hilbert basis theorem. Minimal primes. Localization. Polynomial rings and algebraic sets. Weak Nullstellensatz. Nilradical and Jacobson radical; strong Nullstellensatz. Integral extensions. Prime ideals in integral extensions. Noether Normalization Lemma. Krull dimension; dimension of an affine algebra. Noetherian rings of small dimension, Dedekind domains.

5.4 Reading List

1. M. F. Atiyah and I. G. MacDonald: Introduction to Commutative Algebra, (Addison-Wesley, 1969). 2. D. Eisenbud: Commutative Algebra with a view towards Algebraic Geometry, (Springer GTM, 1995).

6 B2.3 Lie Algebras

6.1 General Prerequisites

A thorough understanding of the material of Part A Linear Algebra, and of basic notions of abstract algebra such as group actions, rings, ideals, quotients *etc.*.

The Michaelmas term Part B course "Introduction to Representation Theory" is recommended. Although the results of that course are not largely not logically needed for the Lie algebras course, the representation theory of Lie algebras mirrors that of groups, and so some familiarity with the concepts of the Michaelmas course is very helpful. Some notions are also introduced in the Part A Group Theory course.

6.2 Overview

Lie Algebras are mathematical objects which, besides being of interest in their own right, elucidate problems in several areas in mathematics. The classification of the finite-dimensional complex Lie algebras is a beautiful piece of applied linear algebra. The aims of this course are to introduce Lie algebras, develop some of the techniques for studying them, and outline the techniques and structures that go into this classification theorem, which shows that semisimple Lie algebras are encoded in finite sets of highly symmetric vectors in a Euclidean vector space known as root systems, which in turn are classified by a kind of graph known as a Dynkin diagram.

6.3 Learning Outcomes

By the end of the course, students will be able to identify the basic classes of Lie algebras - nilpotent, solvable and semisimple, give examples of each, and appreciate the role they play in understanding the structure of Lie algebras. They should be able to use basic notions such as ideals and representations to analyse the structure of Lie algebras, and employ the Cartan criteria. They should also be able to analyse concrete examples of semisimple Lie algebras and identify the associated Dynkin diagram.

Although not all of the key theorems are proved in this course, should they need to, a student who has internalised the techniques used in these lectures should not have too much difficulty filling in these gaps using any of the standard textbooks on the subject.

6.4 Synopsis

- Definition of Lie algebras, small-dimensional examples, some classical groups and their Lie algebras (treated informally). Ideals, subalgebras, homomorphisms, isomorphism theorems.

- Basics of representation theory: $\mathfrak{g}$-modules (or equivalently $\mathfrak{g}$-representations). Irreducible and indecomposable representations, semisimplicity. Composition series and the Jordan-Hölder theorem for modules. Operations on representations: subrepresentations, quotients, Homs and tensor products.

- Composition series for Lie algebras. Short exact sequences and the notion of an extension. Split sequences and semi-direct products. Definition of solvable and nilpotent Lie algebras. A representation in which every element of the Lie algebra acts nilpotently has the trivial representation as its only composition factor. Engel's theorem.

- Representations of solvable Lie algebras over an algebraically closed field of characteristic zero including Lie's theorem. Decomposition of representations of nilpotent

Lie algebras into generalised weight spaces.

- Cartan subalgebras and the Cartan decomposition. Trace forms and Cartan's criterion for solvability. The solvable radical and Cartan's criterion for semisimplicity. Semisimple and simple Lie algebras. The Jordan decomposition. The representations of $\mathfrak{sl}_2$ (*to be done through problem set questions – only the classification of irreducibles is needed for use elsewhere in the course*).

- The Cartan decomposition of a semisimple Lie algebra and the structure of the root system using the representation theory of $\mathfrak{sl}_2$. The abstract root system attached to a semisimple Lie algebra and the reduction of the classification of abstract root systems to the classification of Cartan matrices/Dynkin diagrams. Statement of the classification theorem and informal discussion of the proof of the classification of semisimple Lie algebras.

6.5 Reading List

- *Introduction to Lie algebras*, K. Erdmann, M. Wildon, Springer Undergraduate Mathematics Series. (Available online through the Bodleian at `https://link.springer.com/book/10.1007/1-84628-490-2`.)

- *Introduction to Lie Groups and Lie algebras*, A. Kirillov, Jr. Cambridge Studies in Advanced Mathematics, C.U.P. Available online at `https://www.cambridge.org/core/books/an-introduction-to-lie-groups-and-lie-algebras/98E68056F3EE57686421863E2B0B5DF4`

- *Lie algebras: Theory and algorithms*, Willem A. de Graff, North-Holland Mathematical Library. Available online at `https://www.sciencedirect.com/bookseries/north-holland-mathematical-library/vol/56/suppl/C`

- *Lie algebras of finite and affine type*, R. Carter, Cambridge Studies in Advanced Mathematics, C.U.P. Available online at `https://www.cambridge.org/core/books/lie-algebras-of-finite-and-affine-type/4E6820728C16DC1F812860C974FBB4F6`

- *Lie Groups, Lie Algebras, and Representations*, Brian C. Hall, Graduate Texts in Mathematics, Springer. Available online at `https://link.springer.com/book/10.1007/978-3-319-13467-3`

- *Representation theory: A First Course*, W. Fulton, J. Harris, Graduate Texts in Mathematics, Springer. Available online at `https://www.vlebooks.com/Account/Logon/?returnurl=https%3a%2f%2fwww.vlebooks.com%2fProduct%2fIndex%2f1922982%3fpage%3d0%26startBookmarkId%3d-1`

7 B3.1 Galois Theory

7.1 General Prerequisites

Rings and Modules is essential and Group Theory is recommended.

7.2 Overview

The course starts with a discussion of the classical problem of solving polynomial equations by radicals. This is followed by the classical theory of Galois field extensions, culminating in some of the classical theorems in the subject: the insolubility of the general quintic equation, the classification of finite fields and the irreducibility of the cyclotomic polynomials over the rational numbers.

7.3 Learning Outcomes

Understanding of the relation between symmetries of roots of a polynomial and its solubility in terms of simple algebraic formulae; working knowledge of interesting group actions in a nontrivial context; working knowledge, with applications, of a nontrivial notion of finite group theory (soluble groups); understanding of the relation between algebraic properties of field extensions.

7.4 Synopsis

Solvability of cubic and quartic equations by radicals. Review of algebraic extensions, the Tower Law, Gauss' Lemma and Eisenstein's criterion. Review of groups acting on sets; upper bounds on the size of the Galois group; the theorem of the Primitive Element. Splitting fields and separable extensions. Characterisation of Galois extensions. The fundamental theorem of Galois theory; explicit examples. Solvability by radicals. Normal extensions. Kummer extensions. Techniques for calculating Galois groups: insolubility of certain quintics. Finite fields, the Frobenius automorphism and classification of finite fields. Cyclotomic extensions and the irreducibility of cyclotomic polynomials over the rationals.

7.5 Reading List

1. J. Rotman, *Galois Theory* (Springer-Verlag, NY Inc, 2001/1990).
2. I. Stewart, *Galois Theory* (Chapman and Hall, 2003/1989).
3. D.J.H. Garling, *A Course in Galois Theory* (Cambridge University Press I.N., 1987).
4. Herstein, *Topics in Algebra* (Wiley, 1975).

8 B3.2 Geometry of Surfaces

8.1 General Prerequisites

Part A Topology is recommended. Multidimensional Analysis and Geometry would be useful but not essential. Also, B3.2 is helpful, but not essential, for B3.3 (Algebraic Curves).

8.2 Overview

Different ways of thinking about surfaces (also called two-dimensional manifolds) are introduced in this course: first topological surfaces and then surfaces with extra structures which allow us to make sense of differentiable functions ('smooth surfaces'), holomorphic functions ('Riemann surfaces') and the measurement of lengths and areas ('Riemannian 2-manifolds').

These geometric structures interact in a fundamental way with the topology of the surfaces. A striking example of this is given by the Euler number, which is a manifestly topological quantity, but can be related to the total curvature, which at first glance depends on the geometry of the surface.

The course ends with an introduction to hyperbolic surfaces modelled on the hyperbolic plane, which gives us an example of a non-Euclidean geometry (that is, a geometry which meets all of Euclid's axioms except the axiom of parallels).

8.3 Learning Outcomes

Students will be able to implement the classification of surfaces for simple constructions of topological surfaces such as planar models and connected sums; be able to relate the Euler characteristic to branching data for simple maps of Riemann surfaces; be able to describe the definition and use of Gaussian curvature; know the geodesics and isometries of the hyperbolic plane and their use in geometrical constructions.

8.4 Synopsis

The concept of a topological surface (or 2-manifold); examples, including polygons with pairs of sides identified. Orientation and the Euler characteristic. Classification theorem for compact surfaces (the proof will not be examined).

Riemann surfaces; examples, including the Riemann sphere, the quotient of the complex numbers by a lattice, and double coverings of the Riemann sphere. Holomorphic maps of Riemann surfaces and the Riemann-Hurwitz formula. Elliptic functions.

Smooth surfaces in Euclidean three-space and their first fundamental forms. The concept of a Riemannian 2-manifold; isometries; Gaussian curvature.

Geodesics. The Gauss-Bonnet Theorem (statement of local version and deduction of global version). Critical points of real-valued functions on compact surfaces.

The hyperbolic plane, its isometries and geodesics. Compact hyperbolic surfaces as Riemann surfaces and as surfaces of constant negative curvature.

8.5 Reading List

1. A. Pressley, *Elementary Differential Geometry*, Springer Undergraduate Mathematics Series (Springer-Verlag, 2001). (Chapters 4-8 and 10-11.)

2. G. B. Segal, *Geometry of Surfaces*, Mathematical Institute Notes (1989).

3. R. Earl, *The Local Theory of Curves and Surfaces*, Mathematical Institute Notes (1999).

4. J. McCleary, *Geometry from a Differentiable Viewpoint*, (Cambridge, 1997).

8.6 Further Reading

1. P. A. Firby and C. E. Gardiner, *Surface Topology* (Ellis Horwood, 1991) (Chapters 1-4 and 7).

2. F. Kirwan, *Complex Algebraic Curves*, Student Texts 23 (London Mathematical Society, Cambridge, 1992) (Chapter 5.2 only).

3. B. O'Neill, *Elementary Differential Geometry* (Academic Press, 1997).

4. M. P. do Carmo, *Differential Geometry of Curves and Surfaces* (Dover, 2016)

9 B3.3 Algebraic Curves

9.1 General Prerequisites

Part A Topology. Multidimensional Analysis and Geometry would be useful but not essential. Projective Geometry is recommended. Also, B3.2 (Geometry of Surfaces) is helpful, but not essential.

9.2 Overview

A real algebraic curve is a subset of the plane defined by a polynomial equation $p(x, y) = 0$. The intersection properties of a pair of curves are much better behaved if we extend this picture in two ways: the first is to use polynomials with complex coefficients, the second to extend the curve into the projective plane. In this course projective algebraic curves are studied, using ideas from algebra, from the geometry of surfaces and from complex analysis.

9.3 Learning Outcomes

Students will know the concepts of projective space and curves in the projective plane. They will appreciate the notion of nonsingularity and know some basic features of intersection theory. They will view nonsingular algebraic curves as examples of Riemann surfaces, and be familiar with divisors, meromorphic functions and differentials.

9.4 Synopsis

Projective spaces, homogeneous coordinates, projective transformations.

Algebraic curves in the complex projective plane. Irreducibility, singular and nonsingular points, tangent lines.

Bezout's Theorem (the proof will not be examined). Points of inflection, and normal form of a nonsingular cubic.

Nonsingular algebraic curves as Riemann surfaces. Meromorphic functions, divisors, linear equivalence. Differentials and canonical divisors. The group law on a nonsingular cubic.

The Riemann-Roch Theorem (the proof will not be examined). The geometric genus. Applications.

9.5 Reading List

1. F. Kirwan, *Complex Algebraic Curves*, Student Texts 23 (London Mathematical Society, Cambridge, 1992), Chapters 2-6.
2. W. Fulton, *Algebraic Curves*, 3rd ed., downloadable at `http://www.math.lsa.umich.edu/~wfulton`

10 B3.4 Algebraic Number Theory

10.1 General Prerequisites

Rings and Modules and Number Theory. B3.1 Galois Theory is an essential pre-requisite. All second-year algebra and arithmetic. Students who have not taken Part A Number Theory should read about quadratic residues in, for example, the appendix to Stewart and Tall. This will help with the examples.

10.2 Overview

An introduction to algebraic number theory. The aim is to describe the properties of number fields, but particular emphasis in examples will be placed on quadratic fields, where it is easy to calculate explicitly the properties of some of the objects being considered. In such fields the familiar unique factorisation enjoyed by the integers may fail, and a key objective of the course is to introduce the class group which measures the failure of this property.

10.3 Learning Outcomes

Students will learn about the arithmetic of algebraic number fields. They will learn to prove theorems about integral bases, and about unique factorisation into ideals. They will learn to calculate class numbers, and to use the theory to solve simple Diophantine equations.

10.4 Synopsis

Field extensions, minimum polynomial, algebraic numbers, conjugates, discriminants, Gaussian integers, algebraic integers, integral basis

Examples: quadratic fields

Norm of an algebraic number

Existence of factorisation

Factorisation in $\mathbb{Q}(\sqrt{d})$

Ideals, $\mathbb{Z}$-basis, maximal ideals, prime ideals

Unique factorisation theorem of ideals

Relationship between factorisation of number and of ideals

Norm of an ideal

Ideal classes

Statement of Minkowski convex body theorem

Finiteness of class number

Computations of class number to go on example sheets

10.5 Reading List

I. Stewart and D. Tall, *Algebraic Number Theory and Fermat's Last Theorem* (Third Edition, Peters, 2002).

10.6 Further Reading

D. Marcus, *Number Fields* (Springer-Verlag, New York-Heidelberg, 1977). ISBN 0-387-90279-1.

11 B3.5 Topology and Groups

11.1 General Prerequisites

Part A Topology is essential and Group Theory is recommended.

11.2 Overview

This course introduces the important link between topology and group theory. On the one hand, associated to each space, there is a group, known as its fundamental group. This can be used to solve topological problems using algebraic methods. On the other hand, many results about groups are best proved and understood using topology. For example, presentations of groups, where the group is defined using generators and relations, have a topological interpretation. One of the highlights of the course is the Nielsen-Schreier Theorem, an important, purely algebraic result, which is proved using topological techniques.

11.3 Learning Outcomes

Students will develop a sound understanding of simplicial complexes, cell complexes and their fundamental groups. They will be able to use algebraic methods to analyse topological spaces and compute the fundamental groups of many spaces, including compact surfaces. They will also be able to address questions about groups using topological techniques.

11.4 Synopsis

Homotopic mappings, homotopy equivalence. Simplicial complexes. Simplicial approximation theorem.

The fundamental group of a space. The fundamental group of a circle. Application: the fundamental theorem of algebra. The fundamental groups of spheres.

Free groups. Existence and uniqueness of reduced representatives of group elements. The fundamental group of a graph.

Groups defined by generators and relations (with examples). Tietze transformations.

The free product of two groups. Amalgamated free products.

The Seifert-van Kampen Theorem.

Cell complexes. The fundamental group of a cell complex (with examples). The realization of any finitely presented group as the fundamental group of a finite cell complex.

Covering spaces. Liftings of paths and homotopies. A covering map induces an injection between fundamental groups. The use of covering spaces to determine fundamental groups: the circle again, and real projective n-space. The correspondence between covering spaces and subgroups of the fundamental group. Regular covering spaces and normal subgroups.

Cayley graphs of a group. The relationship between the universal cover of a cell complex, and the Cayley graph of its fundamental group. The Cayley 2-complex of a group.

The Nielsen-Schreier Theorem (every subgroup of a finitely generated free group is free) proved using covering spaces.

11.5 Reading List

1. John Stillwell, *Classical Topology and Combinatorial Group Theory* (Springer-Verlag, 1993).

11.6 Further Reading

1. D. Cohen, *Combinatorial Group Theory: A Topological Approach*, Student Texts 14 (London Mathematical Society, 1989), Chapters 1-7.

2. A. Hatcher, *Algebraic Topology* (CUP, 2001), Chapter. 1.

3. M. Hall, Jr, *The Theory of Groups* (Macmillan, 1959), Chapters. 1-7, 12, 17 .

4. D. L. Johnson, *Presentations of Groups*, Student Texts 15 (Second Edition, London Mathematical Society, Cambridge University Press, 1997). Chapters. 1-5, 10,13.

5. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory* (Dover Publications, 1976). Chapters. 1-4.

12 B4.1 Functional Analysis I

12.1 General Prerequisites

Part A Integration is essential; the only concepts which will be used are the convergence theorems and the theorems of Fubini and Tonelli, and the notions of measurable functions, integrable functions, null sets and L^p spaces. No knowledge is needed of outer measure, or of any particular construction of the integral, or of any proofs. A good working knowledge of Part A Core Analysis (both metric spaces and complex analysis) is expected.

12.2 Overview

The course provides an introduction to the methods of functional analysis.

It builds on core material in analysis and linear algebra studied in Part A. The focus is on normed spaces and Banach spaces; a brief introduction to Hilbert spaces is included, but a systematic study of such spaces and their special features is deferred to B4.2 Functional Analysis II. The techniques and examples studied in the Part B courses Functional Analysis I and II support, in a variety of ways, many advanced courses, in particular in analysis and partial differential equations, as well as having applications in mathematical physics and other areas.

12.3 Learning Outcomes

Students will have a firm knowledge of real and complex normed vector spaces, with their geometric and topological properties. They will be familiar with the notions of completeness, separability and density, will know the properties of a Banach space and important examples, and will be able to prove results relating to the Hahn-Banach Theorem. They will have developed an understanding of the theory of bounded linear operators on a Banach space.

12.4 Synopsis

Brief recall of material from Part A Metric Spaces and Part A Linear Algebra on real and complex normed vector spaces, their geometry and topology and simple examples of completeness. The norm associated with an inner product and its properties. Banach spaces, exemplified by ℓ^p, L^p, $C(K)$, spaces of differentiable functions. Finite-dimensional normed spaces, including equivalence of norms and completeness. Hilbert spaces as a class of Banach spaces having special properties; examples (Euclidean spaces, ℓ^2, L^2), projection theorem, Riesz Representation Theorem.

Density. Approximation of functions, Stone-Weierstrass Theorem. Separable spaces; separability of subspaces.

Bounded linear operators, examples (including integral operators). Continuous linear functionals. Dual spaces. Statement of the Hahn-Banach Theorem; applications, including density of subspaces and embedding of a normed space into its second dual. Adjoint operators.

12.5 Reading List

1. B.P. Rynne and M.A. Youngson, *Linear Functional Analysis* (Springer SUMS, 2nd edition, 2008), Chapters 2, 4, 5.
2. E. Kreyszig, *Introductory Functional Analysis with Applications* (Wiley, revised edition, 1989), Chapters 2, 4.2-4.3, 4.5, 7.1-7.4.

13 B4.2 Functional Analysis II

13.1 General Prerequisites

B4.1 Functional Analysis I is an essential pre-requisite. A4 Integration is also essential; the only concepts which will be used are the convergence theorems and the theorems of Fubini and Tonelli, and the notions of measurable functions, integrable functions, null sets and L^p spaces. No knowledge is needed of outer measure, or of any particular construction of the integral, or of any proofs. A good working knowledge of Part A Core Analysis (both metric spaces and complex analysis) is expected.

13.2 Overview

The course provides further introduction to the methods of functional analysis. It builds on core material in Part A analysis and linear algebra and in Part B B4.1 Functional Analysis I. On one hand, it delves deeper into operator theory on Banach spaces, and on the other, it gives a systematic study of Hilbert spaces, operators on Hilbert spaces and their special features. The techniques and examples studied in the course, together with that in B4.1, support, in a variety of ways, many advanced courses, in particular in analysis and partial differential equations, as well as having applications in mathematical physics and other areas.

13.3 Learning Outcomes

Students will appreciate the role of completeness through the Baire category theorem and its consequences for operators on Banach spaces. They will have a demonstrable knowledge of the properties of a Hilbert space, including orthogonal complements, orthonormal sets, complete orthonormal sets together with related identities and inequalities. They will be familiar with the theory of linear operators on a Banach or Hilbert space, including adjoint operators, compact, self-adjoint and unitary operators with their spectra. They will know the L^2-theory of Fourier series and be aware of the classical theory of Fourier series and other orthogonal expansions.

13.4 Synopsis

Orthogonality, orthogonal complement, closed subspaces.

Linear operators on Hilbert space, adjoint operators. Self-adjoint operators, orthogonal projections, unitary operators.

Baire Category Theorem and its consequences for operators on Banach spaces (Uniform Boundedness, Open Mapping, Inverse Mapping and Closed Graph Theorems). Strong convergence of sequences of operators.

Weak convergence. Weak precompactness of the unit ball (proof for Hilbert spaces).

Orthonormal sets, Pythagoras, Bessel's inequality. Complete orthonormal sets, Parseval. L^2-theory of Fourier series, including completeness of the trigonometric system. Examples of other orthogonal expansions (Legendre, Laguerre, Hermite etc.).

Spectral theory in Banach and Hilbert spaces, in particular spectra of self-adjoint and unitary operators. Spectral theorem for compact self-adjoint operators.

Brief contextual comments on the classical theory of Fourier series and modes of convergence; exposition of failure of pointwise convergence of Fourier series of some continuous functions.

13.5 Reading List

1. B.P. Rynne and M.A. Youngson, *Linear Functional Analysis* (Springer SUMS, 2nd edition, 2008), Chapters 3, 4.4, 6.
2. E. Kreyszig, *Introductory Functional Analysis with Applications* (Wiley, revised edition, 1989), Chapters 3, 4.7-4.9, 4.12-4.13, 9.1-9.2.
3. N. Young, *An Introduction to Hilbert Space* (Cambridge University Press, 1988), Chs 1-7.

13.6 Further Reading

1. E.M. Stein and R. Shakarchi, *Real Analysis: Measure Theory, Integration & Hilbert Spaces* (Princeton Lectures in Analysis III, 2005), Chapter 4.
2. M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol. I. Functional Analysis* (Academic Press, 1980).
3. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (Universitext. Springer, 2011).
4. P. Lax, *Functional Analysis* (Wiley, 2012).

14 B4.3 Distribution Theory

14.1 General Prerequisites

Part A Integration is essential. A good working knowledge of Part A core Analysis is expected. Part A Integral Transforms is desirable but not essential.

14.2 Overview

Distribution theory can be thought of as the completion of differential calculus, just as Lebesgue integration theory can be thought of as the completion of integral calculus. It was created by Laurent Schwartz in the 20th century, as was Lebesgue's integration theory.

In this course we give an introduction to distributions. It builds on core material in analysis and integration studied in Part A. One of the main areas of applications of distributions is to the theory of partial differential equations, and a brief treatment, mainly through examples, is included.

14.3 Learning Outcomes

Students will become acquainted with the basic techniques that in many situations form the starting point for the modern treatment of PDEs.

14.4 Synopsis

Test functions and distributions on RnRn: definitions and examples, Dirac δ- function, approximate identities and constructions using convolution of functions. Density of test functions in Lebesgue spaces. Smooth partitions of unity. [4 lectures]

The calculus of distributions on RnRn : functions as distributions, operations on distributions, adjoint identities, consistency of derivatives, convolution of test functions and distributions. The Fundamental Theorem of Calculus for distributions. Support and singular support of a distribution. Convolution with a compactly supported distribution.

Examples of distributions defined by principal value integrals and finite parts. Examples of distributional boundary values of holomorphic functions defined in a half plane. [8 lectures]

Distributional and weak solutions of PDEs, absolutely continuous functions, Sobolev functions. Examples of fundamental solutions. Weyl's Lemma for distributions. Convolution rules for support and singular support. [4 lectures]

14.5 Reading List

The main recommended book is 1) R.S. Strichartz, A Guide to Distribution Theory and Fourier Transforms (World Scientific, 1994. Reprinted: 2008, 2015) In particular, Chapters 1, 2 and 6.

14.6 Further Reading

2) L.C. Evans, Partial Differential Equations (Amer. Math. Soc. 1998) 3) E.H. Lieb and M. Loss, Analysis (Amer. Math. Soc. 1997) 4) E.M. Stein and R. Shakarchi, Fourier analysis. An introduction (Princeton Univ. Press 2003)

15 B4.4 Fourier Analysis

15.1 General Prerequisites

Distribution Theory and Analysis of PDEs is a pre-requisite.

15.2 Overview

Distribution theory can be thought of as the completion of differential calculus, just as Lebesgue integration theory can be thought of as the completion of integral calculus. It was created by Laurent Schwartz in the 20th century, as was Lebesgue's integration theory.

Distribution theory is a powerful tool that works very well in conjunction with the theory of Fourier transforms. One of the main areas of applications is to the theory of partial differential equations. In this course we give an introduction to these three theories.

15.3 Learning Outcomes

Students will become acquainted with the basic techniques that in many situations form the starting point for the modern treatment of PDEs.

15.4 Synopsis

The Fourier transform on Rn: the Schwartz class SS of test functions on Rn, properties of the Fourier transform on SS, the Fourier transform of a Gaussian and the inversion formula on SS. [4 lectures]

The class of tempered distributions S'S' and their calculus. Fourier transforms of tempered distributions: definitions and examples, convolutions with tempered distributions. The inversion formula on S'S'. Fourier transform in L2L2 and Plancherel's theorem. The Sobolev scale H^s. Elliptic PDEs and Gårding's inequality. [5 lectures]

Fundamental solutions for elliptic PDEs and hypoellipticity. [3 lectures]

The Riemann-Lebesgue lemma, Paley-Wiener theorems, the Poisson summation formula, periodic distributions and Fourier series, the uncertainty principle. [4 lectures]

15.5 Reading List

The main recommended book is

1) R.S. Strichartz, A Guide to Distribution Theory and Fourier Transforms (World Scientific, 1994. Reprinted: 2008, 2015) In particular, Chapters 3-5 and Sections 7.1, 7.2, 7.3 and 7.5.

15.6 Further Reading

2) L.C. Evans, Partial Differential Equations (Amer. Math. Soc. 1998) 3) E.H. Lieb and M. Loss, Analysis (Amer. Math. Soc. 1997) 4) E.M. Stein and R. Shakarchi, Fourier analysis. An introduction (Princeton Univ. Press 2003)

16 B8.1 Probability, Measure and Martingales

16.1 General Prerequisites

The course is self-contained but subsumes both Part A Probability and Part A Integration. It relies strongly on the intuition and knowledge built up in those two courses and both are strongly recommended.

16.2 Overview

Probability is both a fundamental way of viewing the world and a core mathematical discipline. In recent years there has been an explosive growth in the importance of probability in scientific research. Applications range from physics to neuroscience, from genetics to communication networks and, of course, finance.

This course develops the mathematical foundations essential for more advanced courses in probability theory. The first part of the course develops a more sophisticated understanding of measure theory and integration, first seen in Part A Integration. The second part focuses on key probabilistic concepts: independence and conditional expectation. We then introduce discrete time martingales and establish results needed to study their behaviour. This prepares the ground for continuous martingales, studied in B8.2, which are the cornerstone of stochastic calculus.

16.3 Learning Outcomes

The students will learn about measure theory, random variables, independence, expectation and conditional expectation, product measures, filtrations and stopping times, discrete-parameter martingales and their properties and applications.

16.4 Synopsis

Measurable sets, σ-algebras, π-λ systems lemma. Random variables, generated σ-algebras, monotone class theorem. Measures: properties, uniqueness of extension, Caratheodory's Extension Theorem; measure spaces, pushforward measure, product measure.

Independence of events, random variables and σ-algebras, relation to product measures.

The tail σ-algebra, Kolomogorov's 0-1 Law, lim sup and lim inf of a sequence of events, Fatou and reverse Fatou Lemma for sets, Borel-Cantelli Lemmas.

Integration and expectation, review and extension of elementary properties of the integral and convergence theorems [from Part A Integration for the Lebesgue measure on R].

Radon-Nikodym Theorem [without proof], Scheffé's Lemma. Integration on product space, Fubini/Tonelli Theorem. Different modes of convergence and their relations. Markov's and Jensen's inequalities. Lp spaces, Holder's and Minkowski's inequalities, completeness. Uniform integrability, Vitali's convergence theorem.

Conditional expectation: definition, properties, uniquness. Conditional convergence theorems and inequalities, link with uniform integrability. Orthogonal projection in L2, existence of conditional expectation.

Filtrations and stopping times. Examples and properties. σ-algebra associated to a stopping time.

Martingales in discrete time: definition, examples, properties, discrete stochastic integrals. Doob's decomposition theorem. Stopped martingales and Doob's Optional Sampling Theorem. Maximal and Lp Inequalities, Doob's Upcrossing Lemma and Martingale Convergence Theorem. Uniformly integrable martingales, convergence in L1. Backwards martingales and Kolmogorov's Strong Law of Large Numbers.

16.5 Reading List

1. Lecture Notes for the course.
2. D. Williams, *Probability with Martingales*, Cambridge University Press, 1995.

16.6 Further Reading

1. Z. Brzezniak and T. Zastawniak, Basic stochastic processes. A course through exercises. Springer Undergraduate Mathematics Series. (Springer-Verlag London, Ltd., 1999) [more elementary than D. Williams' book, but can provide with a complementary first reading].
2. M. Capinski and E. Kopp, Measure, integral and probability, Springer Undergraduate Mathematics Series. (Springer-Verlag London, Ltd., second edition, 2004).
3. R. Durrett, Probability: Theory and Examples (Second Edition Duxbury Press, Wadsworth Publishing Company, 1996).
4. J. Neveu, Discrete-parameter Martingales (North-Holland, Amsterdam, 1975).

17 B8.2 Continuous Martingales and Stochastic Calculus

17.1 General Prerequisites

B8.1 Probability, Measure and Martingales is a prerequisite. Consequently, Part A Integration and Part A Probability are also prerequisites.

17.2 Overview

Stochastic processes - random phenomena evolving in time - are encountered in many disciplines from biology, physics, geology through to economics and finance. This course focuses on developing the mathematics needed to describe stochastic processes evolving continuously in time and introduces the basic tools of stochastic calculus which are a cornerstone of modern probability theory. The canonical example of such a stochastic process is Brownian motion, also called the Wiener process. This mathematical object was initially proposed by Bachelier, as a model for asset prices, and by Einstein to describe the displacement of a pollen particle in a fluid. The paths of Brownian motion, or of any continuous martingale, are of infinite variation (they are in fact nowhere differentiable and have non-zero quadratic variation) and one of the aims of the course is to define a theory of integration along such paths as well as a suitable version of integration by parts, given by Itô's formula.

17.3 Learning Outcomes

The students will develop an understanding of Brownian motion and continuous martingales in continuous time. They will became familiar with stochastic calculus and in particular be able to use Itô's formula.

17.4 Synopsis

An introduction to stochastic processes in continuous time.

Brownian motion - definition, construction and basic properties, regularity of paths.

Filtrations and stopping times, first hitting times.

Brownian motion - martingale and strong Markov properties, reflection principle.

Martingales - definitions, regularisation and convergence theorems, optional sampling theorem, maximal and Doob's L^p inequalities.

Quadratic variation, local martingales, semimartingales.

Recall of Stieltjes integral.

Stochastic integration and Itô's formula with applications.

17.5 Reading List

Lecture notes will be provided, but there are also many textbooks which cover the course material with a varying degrees of detail/rigour. These include:

1. D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Springer (Revised 3^{rd} ed.), 2001, Chapters 0-4.
2. I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus*, Springer (2^{nd} ed.), 1991, Chapters 1-3.
3. R. Durrett, *Stochastic Calculus: A practical introduction*, CRC Press, 1996. Sections 1.1 - 2.10.

4. F. Klebaner, *Introduction to Stochastic Calculus with Applications*, 3rd edition, Imperial College Press, 2012. Chapters 1, 2, 3.1–3.11, 4.1-4.5, 7.1-7.8, 8.1-8.7.

5. J. M. Steele, *Stochastic Calculus and Financial Applications*, Springer, 2010. Chapters 3 - 8.

6. B. Oksendal, *Stochastic Differential Equations: An introduction with applications*, 6th edition, Springer (Universitext), 2007. Chapters 1 - 3.

7. S. Shreve, *Stochastic calculus for finance*, Vol 2: Continuous-time models, Springer Finance, Springer-Verlag, New York, 2004. Chapters 3 - 4.

18 B8.4 Information Theory

18.1 General Prerequisites

Part A Probability would be very helpful, but not essential.

18.2 Overview

Information theory is a relatively young subject. It played an important role in the rise of the current information/digital/computer age and still motivates much research in diverse fields such as statistics and machine learning, physics, computer science and engineering. Every time you make a phone call, store a file on your computer, query an internet search engine, watch a DVD, stream a movie, listen to a CD or mp3 file, etc., algorithms run that are based on topics we discuss in this course. However, independent of such applications, the underlying mathematical objects arise naturally as soon as one starts to think about "information" in a mathematically rigorous way. In fact, a large part of the course deals with two fundamental questions:

1. How much information is contained in a signal/data/message? (source coding)

2. What are the limits to information transfer over a channel that is subject to noisy perturbations? (channel coding)

18.3 Learning Outcomes

The student will have learned about entropy, mutual information and divergence, their basic properties, how they relate to information transmission. Knowledge of fundamentals of block/symbol/channel coding. Understand the theoretical limits of transmitting information due to noise.

18.4 Synopsis

(Conditional) entropy, mutual information, divergence and their basic properties and inequalities (Fano, Gibbs'). (Strong and weak) typical sequences: the asymptotic equipartition property, and applications to block coding. Symbol codes: Kraft-McMillan, optimality, various symbols codes and their construction and complexity Channel coding: discrete memoryless channels, channel codes/rates/errors, Shannons' noisy channel coding theorem.

18.5 Reading List

1. H. Oberhauser, *B8.4 Information theory* (online lecture notes)
2. T. Cover and J. Thomas, *Elements of Information Theory* (Wiley, 1991), Chapters 1-8, 11.
3. D. MacKay, *Information Theory, Inference, and Learning Algorithms* (Cambridge, 2003)

18.6 Further Reading

1. R. B. Ash, *Information Theory* (Dover, 1990).
2. D. J. A. Welsh, *Codes and Cryptography* (Oxford University Press, 1988), Chapters 1-3, 5.
3. G. Jones and J. M. Jones, *Information and Coding Theory* (Springer, 2000), Chapters 1-5.
4. Y. Suhov & M. Kelbert, *Information Theory and Coding by Example* (Cambridge University Press, 2013), Relevant examples.

19 B8.5 Graph Theory

19.1 General Prerequisites

Part A Graph Theory is recommended.

19.2 Overview

Graphs (abstract networks) are among the simplest mathematical structures, but nevertheless have a very rich and well-developed structural theory. Since graphs arise naturally in many contexts within and outside mathematics, Graph Theory is an important area of mathematics, and also has many applications in other fields such as computer science.

The main aim of the course is to introduce the fundamental ideas of Graph Theory, and some of the basic techniques of combinatorics.

19.3 Learning Outcomes

The student will have developed a basic understanding of the properties of graphs, and an appreciation of the combinatorial methods used to analyze discrete structures.

19.4 Synopsis

Introduction: basic definitions and examples.

Trees and their characterization.

Euler circuits; long paths and cycles.

Vertex colourings: Brooks' theorem, chromatic polynomial.

Edge colourings: Vizing's theorem.

Planar graphs, including Euler's formula, dual graphs.

Maximum flow - minimum cut theorem: applications including Menger's theorem and Hall's theorem.

Tutte's theorem on matchings.

Extremal Problems: Turan's theorem, Zarankiewicz problem, Erdős-Stone theorem.

19.5 Reading List

1. B. Bollobas, *Modern Graph Theory*, Graduate Texts in Mathematics 184 (Springer-Verlag, 1998)

19.6 Further Reading

1. J. A. Bondy and U. S. R. Murty, *Graph Theory: An Advanced Course*, Graduate Texts in Mathematics 244 (Springer-Verlag, 2007).
2. R. Diestel, *Graph Theory*, Graduate Texts in Mathematics 173 (third edition, Springer-Verlag, 2005).
3. D. West, *Introduction to Graph Theory*, Second edition, (Prentice-Hall, 2001).

20 SB3.1 Applied Probability

20.1 General Prerequisites

Part A Probability.

20.2 Overview

This course is intended to show the power and range of probability by considering real examples in which probabilistic modelling is inescapable and useful. Theory will be developed as required to deal with the examples.

20.3 Synopsis

Poisson processes and birth processes. Continuous-time Markov chains. Transition rates, jump chains and holding times. Forward and backward equations. Class structure, hitting times and absorption probabilities. Recurrence and transience. Invariant distributions and limiting behaviour. Time reversal. Renewal theory. Limit theorems: strong law of large numbers, strong law and central limit theorem of renewal theory, elementary renewal theorem, renewal theorem, key renewal theorem. Excess life, inspection paradox.

Applications in areas such as: queues and queueing networks - M/M/s queue, Erlang's formula, queues in tandem and networks of queues, M/G/1 and G/M/1 queues; insurance ruin models; applications in applied sciences.

20.4 Reading List

1. J. R. Norris, *Markov Chains* (Cambridge University Press, 1997).
2. G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes* (3rd edition, Oxford University Press, 2001).
3. G. R. Grimmett and D. R. Stirzaker, *One Thousand Exercises in Probability* (Oxford University Press, 2001).
4. S. M. Ross, *Introduction to Probability Models* (4th edition, Academic Press, 1989).
5. D. R. Stirzaker, *Elementary Probability* (2nd edition, Cambridge University Press, 2003).

21 BEE Mathematical Extended Essay

21.1 Overview

An essay on a mathematical topic may be offered for examination at Part B as a double unit. It is equivalent to a 32-hour lecture course. Generally, students will have 6 hours of supervision distributed over Michaelmas and Hilary terms. In addition there are lectures on writing mathematics and using LaTeX in Michaelmas and Hilary terms. See the lecture list for details.

Students considering offering an essay should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics.* available at:

`https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`

Application Students must apply to the Mathematics Projects Committee for approval of their proposed topic in advance of beginning work on their essay. Proposals should be addressed to the Chair of the Projects Committee, c/o Undergraduate Studies Assistant, Room S0.15, Mathematical Institute and are accepted from the end of Trinity Term. All proposals must be received before 12noon on Friday of Week 0 of Michaelmas Full Term. Note that a BEE essay must have a substantial mathematical content. The application form is available at: `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`

Once a title has been approved, it may only be changed by approval of the Chair of the Projects Committee.

Assessment Each project is independently double-marked, normally by the project supervisor and one other assessor. The two marks are then reconciled to give the overall mark awarded. The reconciliation of marks is overseen by the examiners and follows the department's reconciliation procedure (see `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`).

Submission An electronic copy of your dissertation should be submitted via the Mathematical Institute website to arrive no later than **12 noon on Monday of week 1, Trinity Term 2024**. Further details may be found in the *Guidance Notes on Extended Essays and Dissertations in Mathematics*.

22 BSP Structured Projects

22.1 Overview

Quota Students will be able to choose a project from a menu of six to eight possibilities. Each project has a quota of two to three students.

Students who wish to take BSP **must** register their interest in BSP in the July course registration. After the course registration form has closed, students will be contacted with further details of the projects available for next year. Students will then be informed at a later date whether they have been allocated a place on a BSP project.

22.2 Learning Outcomes

This option is designed to help students understand applications of mathematics to live research problems and to learn some of the necessary techniques. For those who plan to stay on for the MMath or beyond, the course will provide invaluable preliminary training. For those who plan to leave after the BA, it will offer insights into what mathematical research can involve, and training in key skills that will be of long term benefit in any career.

Students will gain experience of:

1. Applications of numerical computation to current research problems.
2. Reading and understanding research papers.
3. Working with new people in new environments.
4. Meeting the expectations of different disciplines.
5. Presenting a well structured written report, using LaTeX.
6. Making an oral presentation to a non-specialist audience.

7. Reading and assessing the work of other students.

8. Independent study and time management.

Students will be expected to:

1. Learn about a current research problem by reading one or more relevant research papers together with appropriate material from textbooks.

2. Carry out the required calculations using Maple, MuPAD or Matlab. Students are not expected to engage in original research but there will be scope for able students to envisage new directions.

3. Write up the problem and their findings in a report that is properly supported with detail, discussion, and good referencing.

4. Undertake peer review.

5. Give an oral presentation to a non-specialist audience.

22.3 Synopsis

In past years projects have included applications to biology, finance, and earth sciences. It is expected that a similar menu of topics, from which students will select one, will be available for 2023-2024.

Teaching

At the beginning of the course students will be given written instructions for their chosen project.

Michaelmas Term

There will be a group meeting with the organiser (Cath Wilkins) at the beginning of MT to set out expectations and deal with queries. The organiser will meet again with students individually at the end of MT. Between these meetings students will read around their chosen topic and take preparatory courses in LaTeX and Matlab, both of which are available from the department and are well documented online. Individual contact with the organiser by email, or if necessary in person, will be encouraged.

Hilary Term

Week 1 Lecture on expectations for the term, and advice on writing up.

Weeks 2 to 8 Students will meet regularly with their specialist supervisor. In addition, each student will meet at least once with the organiser, who will maintain an overview of the student's progress.

Week 10 Submission of written paper.

Easter vacation

Peer review

Trinity Term

Week 1 Oral presentation

Assessment

Students (and tutors) have sometimes expressed doubts about the predictability or reliability of project assessment. We are therefore concerned:

[i.] to make the assessment scheme as transparent as possible both to students and to assessors;

[ii.] that students who produce good project work should be able to achieve equivalent grades to students who write good exam papers.

The mark breakdown will be as follows:

[a.] Written work 75%, of which:

50% of available marks will be for general explanation and discussion of the problem;

50% of available marks will be for mathematical calculations and commentary

[b.] Oral presentation 15%

[c.] Peer review 10%

Note on (c): This may be a new kind of assessment for you. As with journal peer review, the anonymity of both writer and reviewer will be strictly maintained. Each student will be expected to read one other project write-up (from this or previous years) and to make a careful and well explained judgement on it. Credit for this will go to the reviewer, not to the writer, whose work will already have been assessed by examiners in the usual way.

23 BO1.1 History of Mathematics

23.1 Overview

Quota The maximum number of students that can be accepted will be 20. Students should note, however, that numbers are unlikely to reach this level, and so there is little danger of not being accepted onto the course.

23.2 Learning Outcomes

This course is designed to provide the historical background to some of the mathematics familiar to students from A-level and the first four terms of undergraduate study, and looks at a period from approximately the mid-sixteenth century to the end of the nineteenth century. The course will be delivered through 16 lectures in Michaelmas Term, and a reading course consisting of 8 seminars (equivalent to a further 16 lectures) in Hilary Term. Guidance will be given throughout on reading, note-taking, and essay-writing.

Students will gain:

1. an appreciation of university mathematics in its historical context;
2. an enriched understanding of the mathematical content of the topics covered by the course;

3. a broader, multicultural view of mathematics

together with skills in:

1. reading and analysing historical mathematical sources;
2. reading and analysing secondary sources;
3. efficient note-taking;
4. essay-writing (from 1000 to 3000 words);
5. construction of references and bibliographies;
6. oral discussion and presentation.

23.3 Synopsis

Lectures

The Michaelmas Term lectures will cover the following material:

1. Introduction: ancient mathematical knowledge and its transmission to early modern Europe; the development of symbolic notation up to the end of the sixteenth century.
2. Seventeenth century: analytic geometry; the development of calculus; Newton's *Principia*.
3. Eighteenth century: from calculus to analysis; functions, limits, continuity; equations and solvability.
4. Nineteenth century: group theory and abstract algebra; the beginnings of modern analysis; rigorous definitions of real numbers; integration; complex analysis; set theory; linear algebra.

Classes to accompany the lectures will be held in Weeks 3, 5, 6, 7. For each class students will be expected to prepare one piece of written work (1000 words) and one discussion topic. Students will also be expected to present the content of their essays to the whole class.

Reading course

The Hilary Term part of the course is run as a reading course during which we will study a selection of primary texts in some detail, using original sources and secondary literature. Details of the books to be read in HT 2024 will be decided and discussed towards the end of MT 2023. Students will be expected to write three essays (2000 words each) during the first six weeks of term.

Assessment The Michaelmas Term material will be examined in a two-hour written paper during Trinity Term. Candidates will be expected to answer two half-hour questions (commenting on extracts) and one one-hour question (essay). The paper will account for 50% of the marks for the course. The Reading Course will be examined by a 3000-word essay at the end of Hilary Term. The title will be set at the beginning of Week 7 and two copies of the project must be submitted to the Examination Schools by midday on Monday of Week 10; submission of an electronic version of the essay will also be required by the same deadline. The essay will account for 50% of the marks for the course.

23.4 Reading List

1. Jacqueline Stedall, *Mathematics emerging: a sourcebook 1540-1900* (Oxford University Press, 2008).
2. Victor Katz, *A history of mathematics* (brief edition) (Pearson Addison Wesley, 2004), or:
3. Victor Katz, *A history of mathematics: an introduction* (third edition) (Pearson Addison Wesley, 2009).
4. Benjamin Wardhaugh, *How to read historical mathematics* (Princeton, 2010).
5. Jacqueline Stedall, *The history of mathematics: a very short introduction* (Oxford University Press, 2012).

23.5 Further Reading

1. John Fauvel and Jeremy Gray (eds), *The history of mathematics: a reader*, (Macmillan, 1987).
2. June Barrow-Green, Jeremy Gray and Robin J. Wilson, *The history of mathematics : a source-based approach*, vol. I (Mathematical Association of America, 2019).

Further suggestions of additional reading on particular topics will be given throughout the lecture course. Moreover, the intercollegiate classes in MT and the seminars in HT will also serve as a forum in which students will be encouraged to share any interesting reading materials that they have discovered themselves.

24 BOE: Other Mathematical Extended Essay

24.1 Overview

An essay on a topic related to mathematics may be offered for examination at Part B as a double unit. It is equivalent to a 32-hour lecture course. Generally, students will have 6 hours of supervision distributed over Michaelmas and Hilary terms. In addition there are lectures on writing mathematics and using LaTeX in Michaelmas and Hilary terms. See the lecture list for details.

Students considering offering an essay should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics* available at:

`https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`

Application Students must apply to the Mathematics Projects Committee for approval of their proposed topic in advance of beginning work on their essay. Proposals should be addressed to the Chair of the Projects Committee, c/o Undergraduate Studies Administrator, Room S0.15, Mathematical Institute and are accepted from the end of Trinity Term. All proposals must be received before 12noon on Friday of Week 0 of Michaelmas Full Term. The application form is available at `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`.

Once a title has been approved, it may only be changed by approval of the Chair of the Projects Committee.

Assessment Each project is independently double-marked, normally by the project supervisor and one other assessor. The two marks are then reconciled to give the overall mark awarded. The reconciliation of marks is overseen by the examiners and follows the department's reconciliation procedure (see `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`).

Submission An electronic copy of your dissertation should be submitted via the Mathematical Institute website to arrive no later than **12 noon on Monday of week 1, Trinity Term 2024**. Further details may be found in the *Guidance Notes on Extended Essays and Dissertations in Mathematics.*

25 An Introduction to LaTeX

25.1 General Prerequisites

There are no prerequisites. The course is mainly intended for students writing a Part B Extended Essay or a Part C Dissertation but any students are welcome to attend the two lectures given in Michaelmas Term. Note that there is no assessment associated with this course, nor credit for attending the course.

25.2 Overview

This short lecture series provides an introduction to LaTeX.

LaTeX is a markup language, released by Donald Knuth in 1984 and freely sourced, for the professional typesetting of mathematics. (It is based on the earlier TeX released in 1978.) A markup language provides the means for rendering text in various ways - such as bold, italicized or Greek symbols - with the main focus of LaTeX being the rendering of mathematics so that even complicated expressions involving equations, integrals and matrices and images can be professionally typeset.

25.3 Learning Outcomes

Following these introductory lectures, a student should feel comfortable writing their own LaTeX documents, and producing professionally typset mathematics. The learning curve to producing a valid LaTeX document is shallow, and students will further become familiar will some of the principal features of LaTeX such as chapters, item lists, typesetting mathematics, including equations, tables, bibliographies and images. Then, with the aid of a good reference manual, a student should feel comfortable researching out for themselves further features and expanding their LaTeX vocabulary

25.4 Reading List

The Department has a page of LaTeX resources at `https://www.maths.ox.ac.uk/members/it/faqs/latex` which has various free introductory guides to LaTeX.