UNIVERSITY OF OXFORD
Mathematical Institute

BA in MATHEMATICS & PHILOSOPHY
MMath in MATHEMATICS & PHILOSOPHY

SUPPLEMENT TO THE UNDERGRADUATE HANDBOOK – 2016 Matriculation

Preliminary Examination 2016-17
For examination in 2017
SYLLABUS and
SYNOPSES OF LECTURE COURSES

These details can also be found at:
https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/handbooks-synopses

Issued October 2016

Prelims Syllabus and Synopses 2016–2017
Honour School of Mathematics & Philosophy
for examination in 2017

Contents

1 Foreword

Syllabus

The syllabus here is that referred to in the *Examination Regulations 2016* **Special Regulations for Preliminary Examinations in Mathematics and Philosophy**.

Examination Conventions can be found at: http://www.maths.ox.ac.uk/members/students /undergraduate-courses/examinations-assessments/examination-conventions

Synopses

The synopses give some additional detail, and show how the material is split between the different lecture courses. They also include details of recommended reading.

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet, should be addressed to the Academic Administrator (academic.administrator@maths.ox.ac.uk) in the Mathematical Institute.

2 Syllabus

This section contains the Examination Syllabus.

2.1 Mathematics I

Sets: examples including the natural numbers, the integers, the rational numbers, the real numbers; inclusion, union, intersection, power set, ordered pairs and cartesian product of sets. Relations. Definition of an equivalence relation.

The well-ordering property of the natural numbers. Induction as a method of proof, including a proof of the binomial theorem with non-negative integral coefficients.

Maps: composition, restriction, injective (one-to-one), surjective (onto) and invertible maps, images and preimages.

Systems of linear equations. Expression as an augmented matrix (just understood as an array at this point). Elementary Row Operations (EROs). Solutions by row reduction.

Abstract vector spaces: Definition of a vector space over a field (expected examples $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$). Examples of vector spaces: solution space of homogeneous system of equations and differential equations; function spaces; polynomials; $\mathbb{C}$ as an $\mathbb{R}$-vector space; sequence spaces. Subspaces, spanning sets and spans.

Linear independence, definition of a basis, examples. Steinitz exchange lemma, and definition of dimension. Coordinates associated with a basis. Algorithms involving finding a basis of a subspace with EROs. Sums, intersections and direct sums of subspaces. Dimension formula.

Linear transformations: definition and examples including projections. Kernel and image, rank nullity formula.

Algebra of linear transformations. Inverses. Matrix of a linear transformation with respect to a basis. Algebra of matrices. Transformation of a matrix under change of basis. Determining an inverse with EROs. Column space, column rank.

Bilinear forms. Positive definite symmetric bilinear forms. Inner Product Spaces. Examples: $\mathbb{R}^n$ with dot product, function spaces. Comment on (positive definite) Hermitian forms. Cauchy-Schwarz inequality. Distance and angle. Transpose of a matrix. Orthogonal matrices.

Introduction to determinant of a square matrix: existence and uniqueness and relation to volume. Proof of existence by induction. Basic properties, computation by row operations.

Determinants and linear transformations: multiplicativity of the determinant, definition of the determinant of a linear transformation. Invertibility and the determinant. Permutation matrices and explicit formula for the determinant deduced from properties of determinant.

Eigenvectors and eigenvalues, the characteristic polynomial. Trace. Proof that eigenspaces form a direct sum. Examples. Discussion of diagonalisation. Geometric and algebraic multiplicity of eigenvalues.

Gram-Schmidt procedure.

Spectral theorem for real symmetric matrices. Matrix realisation of bilinear maps given a basis and application to orthogonal transformation of quadrics into normal form. Statement of classification of orthogonal transformations.

Axioms for a group and for an Abelian group. Examples including geometric symmetry groups, matrix groups (GL_n, SL_n, O_n, SO_n, U_n), cyclic groups. Products of groups.

Permutations of a finite set under composition. Cycles and cycle notation. Order. Transpositions; every permutation may be expressed as a product of transpositions. The parity of a permutation is well-defined via determinants. Conjugacy in permutation groups.

Subgroups; examples. Intersections. The subgroup generated by a subset of a group. A subgroup of a cyclic group is cyclic. Connection with hcf and lcm. Bezout's Lemma.

Recap on equivalence relations including congruence mod n and conjugacy in a group. Proof that equivalence classes partition a set. Cosets and Lagrange's Theorem; examples. The order of an element. Fermat's Little Theorem.

Isomorphisms. Groups up to isomorphism of order 8 (stated without proof). Homomorphisms of groups. Kernels. Images. Normal subgroups. Quotient groups. First Isomorphism Theorem. Simple examples determining all homomorphisms between groups.

Group actions; examples. Definition of orbits and stabilizers. Transitivity. Orbits partition the set. Stabilizers are subgroups.

Orbit-stabilizer Theorem. Examples and applications including Cauchy's Theorem and to conjugacy classes. Orbit-counting formula.

The representation $G \to \mathrm{Sym}(S)$ associated with an action of G on S. Cayley's Theorem. Symmetry groups of the tetrahedron and cube.

2.2 Mathematics II

Real numbers: arithmetic, ordering, suprema, infima; real numbers as a complete ordered field. Countable sets. The rational numbers are countable. The real numbers are uncountable.

The complex number system. The Argand diagram; modulus and argument. De Moivre's Theorem, polar form, the triangle inequality. Statement of the Fundamental Theorem of Algebra. Roots of unity. De Moivre's Theorem. Simple transformations in the complex plane. Polar form, with applications.

Sequences of (real or complex) numbers. Limits of sequences of numbers; the algebra of limits. Order notation.

Subsequences; every subsequence of a convergent sequence converges to the same limit. Bounded monotone sequences converge. Bolzano–Weierstrass Theorem. Cauchy's convergence criterion. Limit point of a subset of the line or plane.

Series of (real or complex) numbers. Convergence of series. Simple examples to include geometric progressions and power series. Alternating series test, absolute convergence, comparison test, ratio test, integral test.

Power series, radius of convergence, important examples to include defintions of relationships between exponential, trigonometric functions and hyperbolic functions.

Continuous functions of a single real or complex variable. Definition of continuity of real valued functions of several variables.

The algebra of continuous functions. A continuous real-valued function on a closed bounded interval is bounded, achieves its bounds and is uniformly continuous. Intermediate Value Theorem. Inverse Function Theorem for continuous strictly monotonic functions.

Sequences and series of functions. The uniform limit of a sequence of continuous functions is continuous. Weierstrass's M-test. Continuity of functions defined by power series.

Definition of derivative of a function of a single real variable. The algebra of differentiable functions. Rolle's Theorem. Mean Value Theorem. Cauchy's (Generalized) Mean Value Theorem. L'Hôpital's Formula. Taylor's expansion with remainder in Lagrange's form. Binomial theorem with arbitrary index.

Step functions and their integrals. The integral of a continuous function on a closed bounded interval. Properties of the integral including linearity and the interchange of integral and limit for a uniform limit of continuous functions on a bounded interval. The Mean Value Theorem for Integrals. The Fundamental Theorem of Calculus; integration by parts and substitution.

Term-by-term differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly).

2.3 Mathematics III(P)

The syllabus for this paper is as for Paper III in the Preliminary Examination in Mathematics with the omission of Statistics.

General linear homogeneous ODEs: integrating factor for first order linear ODEs, second solution when one solution is known for second order linear ODEs. First and second order linear ODEs with constant coefficients. General solution of linear inhomogeneous ODE as particular solution plus solution of homogeneous equation. Simple examples of finding particular integrals by guesswork.

Partial derivatives. Second order derivatives and statement of condition for equality of mixed partial derivatives. Chain rule, change of variable, including planar polar coordinates. Solving some simple partial differential equations (e.g. $f_{xy} = 0$, $f_x = f_y$).

Parametric representation of curves, tangents. Arc length. Line integrals.

Jacobians with examples including plane polar coordinates. Some simple double integrals calculating area and also $\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$.

Simple examples of surfaces, especially as level sets. Gradient vector; normal to surface; directional derivative; $\int_A^B \nabla\phi \cdot d\mathbf{r} = \phi(B) - \phi(A)$.

Taylor's Theorem for a function of two variables (statement only). Critical points and classification using directional derivatives and Taylor's theorem. Informal (geometrical) treatment of Lagrange multipliers.

Sample space, algebra of events, probability measure. Permutations and combinations, sampling with or without replacement. Conditional probability, partitions of the sample space, theorem of total probability, Bayes' Theorem. Independence.

Discrete random variables, probability mass functions, examples: Bernoulli, binomial, Poisson, geometric. Expectation: mean and variance. Joint distributions of several discrete random variables. Marginal and conditional distributions. Independence. Conditional expectation, theorem of total probability for expectations. Expectations of functions of more than one discrete random variable, covariance, variance of a sum of dependent discrete random variables.

Solution of first and second order linear difference equations. Random walks (finite state space only).

Probability generating functions, use in calculating expectations. Random sample, sums of independent random variables, random sums. Chebyshev's inequality, Weak Law of Large Numbers.

Continuous random variables, cumulative distribution functions, probability density functions, examples: uniform, exponential, gamma, normal. Expectation: mean and variance. Functions of a single continuous random variable. Joint probability density functions of several continuous random variables (rectangular regions only). Marginal distributions. Independence. Expectations of functions of jointly continuous random variables, covariance, variance of a sum of dependent jointly continuous random variables.

Synopses of Lectures

3 Mathematics I

3.1 Introductory Courses

There are two short introductory courses within the first two weeks of Michaelmas term to help students adjust to University Mathematics. These are *Introduction to University Level Mathematics* and *Introduction to Complex Numbers*.

3.1.1 Introduction to University Level Mathematics — Prof. Alan Lauder — 8 MT

There will be 8 introductory lectures in the first two weeks of Michaelmas term.

Overview

Prior to arrival, undergraduates are encouraged to read Professor Batty's study guide "How do undergraduates do Mathematics?" https://www.maths.ox.ac.uk/system/files/attachments/study_public_1.pdf

The purpose of these introductory lectures is to establish some of the basic language and notation of university mathematics, and to introduce the elements of (naïve) set theory and the nature of formal proof.

Learning Outcomes

Students should:

(i) have the ability to describe, manipulate, and prove results about sets and functions using standard mathematical notation;

(ii) know and be able to use simple relations;

(iii) develop sound reasoning skills;

(iv) have the ability to follow and to construct simple proofs, including proofs by mathematical induction (including strong induction, minimal counterexample) and proofs by contradiction;

(v) learn how to write mathematics.

Synopsis

The natural numbers and their ordering. Induction as a method of proof, including a proof of the binomial theorem with non-negative integral coefficients.

Sets. Examples including $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and intervals in $\mathbb{R}$. Inclusion, union, intersection, power set, ordered pairs and cartesian product of sets.

Relations. Definition of an equivalence relation. Examples. Functions: composition, restriction; injective (one-to-one), surjective (onto) and invertible functions; images and preimages.

Writing mathematics. The language of mathematical reasoning; quantifiers: "for all", "there exists". Formulation of mathematical statements with examples.

Proofs and refutations: standard techniques for constructing proofs; counter-examples. Example of proof by contradiction and more on proof by induction.

Problem-solving in mathematics: experimentation, conjecture, confirmation, followed by explaining the solution precisely.

Reading

1. C. J. K. Batty, *How do undergraduates do Mathematics?*, (Mathematical Institute Study Guide, 1994) `https://www.maths.ox.ac.uk/system/files/attachments/study_public_1.pdf`.

Further Reading

1. G. Pólya. *How to solve it: a new aspect of mathematical method* (1945, New edition 2014 with a foreword by John Conway, Princeton University Press).
2. G. C. Smith, *Introductory Mathematics: Algebra and Analysis*, (Springer-Verlag, London, 1998), Chapters 1 and 2.
3. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis*, (Wiley, New York, Fourth Edition, 2011), Chapter 1 and Appendices A and B.
4. C. Plumpton, E. Shipton, R. L. Perry, *Proof*, (MacMillan, London, 1984).
5. R. B. J. T. Allenby, *Numbers and Proofs*, (Butterworth-Heinemann, London, 1997).
6. R. A. Earl, *Bridging Material on Induction*, (Mathematics Department website).

3.1.2 Introduction to Complex Numbers — Dr Peter Neumann — 2 MT

This course of two lectures will run in the first week of Michaelmas Term.

Generally, students should not expect a tutorial to support this short course. Solutions to the problem sheet will be posted on Monday of Week 2 and students are asked to mark their own problems and notify their tutor.

Overview

This course aims to give all students a common background in complex numbers.

Learning Outcomes

Students will be able to:

(i) manipulate complex numbers with confidence;

(ii) understand geometrically their representation on the Argand diagram, including the nth roots of unity;

(iii) know the polar representation form and be able to apply it.

Synopsis

Basic arithmetic of complex numbers, the Argand diagram; modulus and argument of a complex number. Statement of the Fundamental Theorem of Algebra. Roots of unity. De Moivre's Theorem. Simple transformations in the complex plane. Polar form $re^{i\theta}$, with applications.

Reading

1. R. A. Earl, *Complex numbers* (https://www.maths.ox.ac.uk/study-here/undergraduate-study/bridging-gap
2. D. W. Jordan & P Smith, *Mathematical Techniques* (Oxford University Press, Oxford, 2002), Ch. 6.

3.2 Linear Algebra I — Dr Peter Neumann — 14 MT

Overview

Linear algebra pervades and is fundamental to algebra, geometry, analysis, applied mathematics, statistics, and indeed most of mathematics. This course lays the foundations, concentrating mainly on vector spaces and matrices over the real and complex number systems. The course begins with examples focussed on $\mathbb{R}^2$ and $\mathbb{R}^3$, and gradually becomes more abstract. The course also introduces the idea of an inner product, with which angle and distance can be introduced into a vector space.

Learning Outcomes

Students will:

(i) understand the notions of a vector space, a subspace, linear dependence and independence, spanning sets and bases within the familiar setting of $\mathbb{R}^2$ and $\mathbb{R}^3$;

(ii) understand and be able to use the abstract notions of a general vector space, a subspace, linear dependence and independence, spanning sets and bases and be able to formally prove results related to these concepts;

(iii) have an understanding of matrices and of their applications to the algorithmic solution of systems of linear equations and to their representation of linear maps between vector spaces.

Synopsis

Systems of linear equations. Expression as an augmented matrix (understood simply as an array at this point). Elementary Row Operations (EROs). Solutions by row reduction.

Abstract vector spaces: Definition of a vector space over a field (expected examples $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$). Examples of vector spaces: solution space of homogeneous systems of equations and differential equations; function spaces; polynomials; $\mathbb{C}$ as an $\mathbb{R}$-vector space; sequence spaces. Subspaces, spanning sets and spans. (Emphasis on concrete examples, with deduction of properties from axioms set as problems).

Linear independence, definition of a basis, examples. Steinitz exchange lemma, and definition of dimension. Coordinates associated with a basis. Algorithms involving EROs to find a basis of a subspace.

Sums, intersections and direct sums of subspaces. Dimension formula.

Linear transformations: definition and examples including projections. Kernel and image, rank–nullity formula.

Algebra of linear transformations. Inverses. Matrix of a linear transformation with respect to a basis. Algebra of matrices. Transformation of a matrix under change of basis. Determining an inverse with EROs. Column space, column rank.

Bilinear forms. Positive definite symmetric bilinear forms. Inner Product Spaces. Examples: $\mathbb{R}^n$ with dot product, function spaces. Comment on (positive definite) Hermitian forms. Cauchy–Schwarz inequality. Distance and angle. Transpose of a matrix. Orthogonal matrices.

Reading

1. T. S. Blyth and E. F. Robertson, *Basic Linear Algebra* (Springer, London, 1998).

2. R. Kaye and R. Wilson, *Linear Algebra* (OUP, 1998), Chapters 1–5 and 8. [More advanced but useful on bilinear forms and inner product spaces.]

Alternative and Further Reading

1. C. W. Curtis, *Linear Algebra – An Introductory Approach* (Springer, London, 4th edition, reprinted 1994).

2. R. B. J. T. Allenby, *Linear Algebra* (Arnold, London, 1995).

3. D. A. Towers, *A Guide to Linear Algebra* (Macmillan, Basingstoke, 1988).

4. D. T. Finkbeiner, *Elements of Linear Algebra* (Freeman, London, 1972). [Out of print, but available in many libraries.]

5. B. Seymour Lipschutz, Marc Lipson, *Linear Algebra* (McGraw Hill, London, Third Edition, 2001).

3.3 Linear Algebra II — Prof. Alan Lauder — 8HT

Learning Outcomes

Students will:

(i) understand the elementary theory of determinants;

(ii) understand the beginnings of the theory of eigenvectors and eigenvalues and appreciate the applications of diagonalizability.

(iii) understand the Spectral Theory for real symmetric matrices, and appreciate the geometric importance of an orthogonal change of variable.

Synopsis

Introduction to determinant of a square matrix: existence and uniqueness. Proof of existence by induction. Proof of uniqueness by deriving explicit formula from the properties of the determinant. Permutation matrices. (No general discussion of permutations). Basic properties of determinant, relation to volume. Multiplicativity of the determinant, computation by row operations.

Determinants and linear transformations: definition of the determinant of a linear transformation, multiplicativity, invertibility and the determinant.

Eigenvectors and eigenvalues, the characteristic polynomial, trace. Eigenvectors for distinct eigenvalues are linearly independent. Discussion of diagonalisation. Examples. Eigenspaces, geometric and algebraic multiplicity of eigenvalues. Eigenspaces form a direct sum.

Gram-Schmidt procedure. Spectral theorem for real symmetric matrices. Quadratic forms and real symmetric matrices. Application of the spectral theorem to putting quadrics into normal form by orthogonal transformations and translations. Statement of classification of orthogonal transformations.

Reading

1. T. S. Blyth and E. F. Robertson, *Basic Linear Algebra* (Springer, London, 2nd edition 2002).

2. C. W. Curtis, *Linear Algebra – An Introductory Approach* (Springer, New York, 4th edition, reprinted 1994).

3. R. B. J. T. Allenby, *Linear Algebra* (Arnold, London, 1995).

4. D. A. Towers, *A Guide to Linear Algebra* (Macmillan, Basingstoke 1988).

5. S. Lang, *Linear Algebra* (Springer, London, Third Edition, 1987).

3.4 Groups and Group Actions — Dr Vicky Neale — 8 HT and 8 TT

Overview

Abstract algebra evolved in the twentieth century out of nineteenth century discoveries in algebra, number theory and geometry. It is a highly developed example of the power of generalisation and axiomatisation in mathematics. The *group* is an important first example of an abstract, algebraic structure and groups permeate much of mathematics particularly where there is an aspect of symmetry involved. Moving on from examples and the theory of groups, we will also see how groups *act* on sets (e.g. permutations on sets, matrix groups on vectors) and apply these results to several geometric examples and more widely.

Learning Outcomes

Students will appreciate the value of abstraction and meet many examples of groups and group actions from around mathematics. Beyond theoretic aspects of group theory students will also see the value of these methods in the generality of the approach and also to otherwise intractable counting problems.

Synopsis

HT (8 lectures)

Axioms for a group and for an Abelian group. Examples including geometric symmetry groups, matrix groups (GL_n, SL_n, O_n, SO_n, U_n), cyclic groups. Products of groups.

Permutations of a finite set under composition. Cycles and cycle notation. Order. Transpositions; every permutation may be expressed as a product of transpositions. The parity of a permutation is well-defined via determinants. Conjugacy in permutation groups.

Subgroups; examples. Intersections. The subgroup generated by a subset of a group. A subgroup of a cyclic group is cyclic. Connection with hcf and lcm. Bezout's Lemma.

Recap on equivalence relations including congruence mod n and conjugacy in a group. Proof that equivalence classes partition a set. Cosets and Lagrange's Theorem; examples. The order of an element. Fermat's Little Theorem.

TT (8 Lectures)

Isomorphisms, examples. Groups up to isomorphism of order 8 (stated without proof). Homomorphisms of groups with motivating examples. Kernels. Images. Normal subgroups. Quotient groups; examples. First Isomorphism Theorem. Simple examples determining all homomorphisms between groups.

Group actions; examples. Definition of orbits and stabilizers. Transitivity. Orbits partition the set. Stabilizers are subgroups.

Orbit-stabilizer Theorem. Examples and applications including Cauchy's Theorem and to conjugacy classes.

Orbit-counting formula. Examples.

The representation $G \to \mathrm{Sym}(S)$ associated with an action of G on S. Cayley's Theorem. Symmetry groups of the tetrahedron and cube.

Reading

1. M. A. Armstrong *Groups and Symmetry* (Springer, 1997)

Alternative Reading

1. R. B. J. T. Allenby, *Rings, Fields and Groups*, (Second revised edition, Elsevier, 1991)
2. Peter J. Cameron, *Introduction to Algebra*, (Second edition, Oxford University Press, 2007).
3. John B. Fraleigh, *A First Course in Abstract Algebra* (Seventh edition, Pearson, 2013).
4. W. Keith Nicholson, *Introduction to Abstract Algebra* (Fourth edition, John Wiley, 2012).
5. Joseph J. Rotman, *A First Course in Abstract Algebra* (Third edition, Pearson, 2005).
6. Joseph Gallian, *Contemporary Abstract Algebra* (8th international edition, Brooks/Cole, 2012).
7. Nathan Carter, *Visual Group Theory* (MAA Problem Book Series, 2009).

4 Mathematics II

4.1 Analysis I: Sequences and Series — Prof. Frances Kirwan — 15 MT

Overview

In these lectures we study the real and complex numbers, and study their properties, particularly completeness; define and study limits of sequences, convergence of series, and power series.

Learning Outcomes

Student will have:

(i) an ability to work within an axiomatic framework;

(ii) a detailed understanding of how Cauchy's criterion for the convergence of real and complex sequences and series follows from the completeness axiom for $\mathbb{R}$, and the ability to explain the steps in standard mathematical notation;

(iii) knowledge of some simple techniques for testing the convergence of sequences and series, and confidence in applying them;

(iv) familiarity with a variety of well-known sequences and series, with a developing intuition about the behaviour of new ones;

(v) an understanding of how the elementary functions can be defined by power series, with an ability to deduce some of their easier properties.

Synopsis

Real numbers: arithmetic, ordering, suprema, infima; the real numbers as a complete ordered field. Definition of a countable set. The countability of the rational numbers. The reals are uncountable. The complex number system. The triangle inequality.

Sequences of real or complex numbers. Definition of a limit of a sequence of numbers. Limits and inequalities. The algebra of limits. Order notation: O, o.

Subsequences; a proof that every subsequence of a convergent sequence converges to the same limit; bounded monotone sequences converge. Bolzano–Weierstrass Theorem. Cauchy's convergence criterion.

Series of real or complex numbers. Convergence of series. Simple examples to include geometric progressions and some power series. Absolute convergence, Comparison Test, Ratio Test, Integral Test. Alternating Series Test.

Power series, radius of convergence. Examples to include definition of and relationships between exponential, trigonometric functions and hyperbolic functions.

Reading

1. Lara Alcock, *How to Think About Analysis* (OUP, 2014) ISBN 9780198723530
2. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis* (Wiley, Third Edition, 2000), Chapters 2, 3, 9.1, 9.2.
3. R. P. Burn, *Numbers and Functions, Steps into Analysis* (Cambridge University Press, 2000), Chapters 2–6. [This is a book of problems and answers, a DIY course in analysis.]
4. J. M. Howie, *Real Analysis*, Springer Undergraduate Texts in Mathematics Series (Springer, 2001) ISBN 1-85233-314-6.

Alternative Reading

The first four books take a slightly gentler approach to the material in the syllabus, whereas the last two cover it in greater depth and contain some more advanced material.

1. Mary Hart, *A Guide to Analysis* (MacMillan, 1990), Chapter 2.
2. J. C. Burkill, *A First Course In Mathematical Analysis* (Cambridge University Press, 1962), Chapters 1, 2 and 5.
3. Victor Bryant, *Yet Another Introduction to Analysis* (Cambridge University Press, 1990), Chapters 1 and 2.
4. G.C. Smith, *Introductory Mathematics: Algebra and Analysis* (Springer-Verlag, 1998), Chapter 3 (introducing complex numbers).
5. Michael Spivak, *Calculus* (Benjamin, 1967), Parts I, IV, and V (for a construction of the real numbers).
6. Brian S. Thomson, Judith B. Bruckner, Andrew M. Bruckner, *Elementary Analysis* (Prentice Hall, 2001), Chapters 1–4.

4.2 Analysis II: Continuity and Differentiability — Prof. Hilary Priestley — 16 HT

Overview

In this term's lectures, we study continuity of functions of a real or complex variable, and differentiability of functions of a real variable.

Learning Outcomes

At the end of the course students will be able to apply limiting properties to describe and prove continuity and differentiability conditions for real and complex functions. They will be able to prove important theorems, such as the Intermediate Value Theorem, Rolle's Theorem and Mean Value Theorem, and will continue the study of power series and their convergence.

Synopsis

Definition of the function limit. Examples and counter examples to illustrate when $\lim_{x\to a} f(x) = f(a)$ (and when it doesn't). Definition of continuity of functions on subsets of $\mathbb{R}$ and $\mathbb{C}$ in terms of ε and δ. Continuity of real valued functions of several variables.The algebra of continuous functions; examples, including polynomials. Continuous functions on closed bounded intervals: boundedness, maxima and minima, uniform continuity. Intermediate Value Theorem. Inverse Function Theorem for continuous strictly monotone functions.

Sequences and series of functions. Uniform limit of a sequence of continuous functions is continuous. Weierstrass's M-test for uniformly convergent series of functions. Continuity of functions defined by power series.

Definition of the derivative of a function of a real variable. Algebra of derivatives, examples to include polynomials and inverse functions. The derivative of a function defined by a power series is given by the derived series (proof not examinable). Vanishing of the derivative at a local maximum or minimum. Rolle's Theorem. Mean Value Theorem with simple applications: constant and monotone functions. Cauchy's (Generalized) Mean Value Theorem and L'Hôpital's Formula. Taylor's Theorem with remainder in Lagrange's form; examples of Taylor's Theorem to include the binomial expansion with arbitrary index.

Reading

1. Robert G. Bartle, Donald R. Sherbert, *Introduction to Real Analysis* (Wiley, Fourth Edition, 2011), Chapters 4–8.
2. R. P. Burn, *Numbers and Functions, Steps into Analysis* (Cambridge University Press, Second Edition, 2000). [This is a book of problems and answers, a DIY course in analysis]. Chapters 6–9, 12.
3. Walter Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, Third Edition, 1976). Chapters 4,5,7.
4. J. M. Howie, *Real Analysis,* Springer Undergraduate Texts in Mathematics Series (Springer, 2001), ISBN 1-85233-314-6.

Alternative Reading

1. Mary Hart, *A Guide to Analysis* (MacMillan, Second Edition, 2001), Chapters 4,5.
2. J. C. Burkill, *A First Course in Mathematical Analysis* (Cambridge University Press, 1962), Chapters 3, 4, and 6.

3. K. G. Binmore, *Mathematical Analysis* A Straightforward Approach, (Cambridge University Press, Second Edition, 1982), Chapters 7–12, 14–16.

4. Victor Bryant, *Yet Another Introduction to Analysis* (Cambridge University Press, 1990), Chapters 3 and 4.

5. M. Spivak, *Calculus* (Publish or Perish, Fourth Edition, 2008), Part III.

6. Brian S. Thomson, Judith B. Bruckner, Andrew M. Bruckner, *Elementary Real Analysis* (classicalrealanalysis.com, Second Edition, 2008), Chapters 5–10 [Free Download].

4.3 Analysis III: Integration — Prof. Ben Green — 8 TT

Overview

In these lectures we define a simple integral and study its properties; prove the Mean Value Theorem for Integrals and the Fundamental Theorem of Calculus. This gives us the tools to justify term-by-term differentiation of power series and deduce the elementary properties of the trigonometric functions.

Learning Outcomes

At the end of the course students will be familiar with the construction of an integral from fundamental principles, including important theorems. They will know when it is possible to integrate or differentiate term-by-term and be able to apply this to, for example, trigonometric series.

Synopsis

Step functions, their integral, basic properties. Lower and upper integrals of bounded functions on bounded intervals. Definition of Riemann integrable functions.

The application of uniform continuity to show that continuous functions are Riemann integrable on closed bounded intervals; bounded continuous functions are Riemann integrable on bounded intervals.

Elementary properties of Riemann integrals: positivity, linearity, subdivision of the interval. The Mean Value Theorem for Integrals. The Fundamental Theorem of Calculus; linearity of the integral, integration by parts and by substitution.

The interchange of integral and limit for a uniform limit of continuous functions on a bounded interval. Term-by-term integration and differentiation of a (real) power series (interchanging limit and derivative for a series of functions where the derivatives converge uniformly).

Reading

Lecture notes will be provided

1. H. A. Priestley, *Introduction to Integration* (Oxford Science Publications, 1997)
2. W. Rudin, *Principles of Mathematical Analysis*, (McGraw-Hill, Third Edition, 1976).

Both of these books contain more material than is in the course.

5 Mathematics III(P)

5.1 Introductory Calculus — Dr Cath Wilkins — 16 MT

Overview

These lectures are designed to give students a gentle introduction to applied mathematics in their first term at Oxford, allowing time for both students and tutors to work on developing and polishing the skills necessary for the course. It will have an 'A-level' feel to it, helping in the transition from school to university. The emphasis will be on developing skills and familiarity with ideas using straightforward examples.

Learning Outcomes

At the end of the course, students will be able to solve a range of ordinary differential equations (ODEs). They will also be able to evaluate partial derivatives and use them in a variety of applications.

Synopsis

General linear homogeneous ODEs: integrating factor for first order linear ODEs, second solution when one solution is known for second order linear ODEs. First and second order linear ODEs with constant coefficients. General solution of linear inhomogeneous ODE as particular solution plus solution of homogeneous equation. Simple examples of finding particular integrals by guesswork. [4]

Introduction to partial derivatives. Second order derivatives and statement of condition for equality of mixed partial derivatives. Chain rule, change of variable, including planar polar coordinates. Solving some simple partial differential equations (e.g. $f_{xy} = 0$, $f_x = f_y$). [3.5]

Parametric representation of curves, tangents. Arc length. Line integrals. [1]

Jacobians with examples including plane polar coordinates. Some simple double integrals calculating area and also $\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$. [2]

Simple examples of surfaces, especially as level sets. Gradient vector; normal to surface; directional derivative; $\int_A^B \nabla\phi \cdot d\mathbf{r} = \phi(B) - \phi(A)$.[2]

Taylor's Theorem for a function of two variables (statement only). Critical points and classification using directional derivatives and Taylor's theorem. Informal (geometrical) treatment of Lagrange multipliers.[3.5]

Reading

M. L. Boas, *Mathematical Methods in the Physical Sciences* (Wiley, 3rd Edition, 2005).

D. W. Jordan & P. Smith, *Mathematical Techniques* (Oxford University Press, 3rd Edition, 2003).

E. Kreyszig, *Advanced Engineering Mathematics* (Wiley, 10th Edition, 2011).

K. A. Stroud, *Advanced Engineering Mathematics* (Palgrave Macmillan, 5th Edition, 2011).

5.2 Probability — Prof. James Martin — 16 MT

Overview

An understanding of random phenomena is becoming increasingly important in today's world within social and political sciences, finance, life sciences and many other fields. The aim of this introduction to probability is to develop the concept of chance in a mathematical framework. Random variables are introduced, with examples involving most of the common distributions.

Learning Outcomes

Students should have a knowledge and understanding of basic probability concepts, including conditional probability. They should know what is meant by a random variable, and have met the common distributions. They should understand the concepts of expectation and variance of a random variable. A key concept is that of independence which will be introduced for events and random variables.

Synopsis

Motivation, relative frequency, chance. Sample space, algebra of events, probability measure. Permutations and combinations, sampling with or without replacement. Conditional probability, partitions of the sample space, theorem of total probability, Bayes' Theorem. Independence.

Discrete random variables, probability mass functions, examples: Bernoulli, binomial, Poisson, geometric. Expectation: mean and variance. Joint distributions of several discrete random variables. Marginal and conditional distributions. Independence. Conditional expectation, theorem of total probability for expectations. Expectations of functions of more than one discrete random variable, covariance, variance of a sum of dependent discrete random variables.

Solution of first and second order linear difference equations. Random walks (finite state space only).

Probability generating functions, use in calculating expectations. Random sample, sums of independent random variables, random sums. Markov's inequality, Chebyshev's inequality, Weak Law of Large Numbers.

Continuous random variables, cumulative distribution functions, probability density functions, examples: uniform, exponential, gamma, normal. Expectation: mean and variance. Functions of a single continuous random variable. Joint probability density functions of several continuous random variables (rectangular regions only). Marginal distributions. Independence. Expectations of functions of jointly continuous random variables, covariance, variance of a sum of dependent jointly continuous random variables.

Reading

1. G. R. Grimmett and D. J. A. Welsh, *Probability: An Introduction* (Oxford University Press, 1986), Chapters 1–4, 5.1–5.4, 5.6, 6.1, 6.2, 6.3 (parts of), 7.1–7.3, 10.4.

2. J. Pitman, *Probability* (Springer-Verlag, 1993).

3. S. Ross, *A First Course In Probability* (Prentice-Hall, 1994).

4. D. Stirzaker, *Elementary Probability* (Cambridge University Press, 1994), Chapters 1–4, 5.1–5.6, 6.1–6.3, 7.1, 7.2, 7.4, 8.1, 8.3, 8.5 (excluding the joint generating function).