The geometric PDEs of general relativity

Richard Schoen

Stanford University

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Plan of Lecture

The lecture will have three parts:

Part 1: Introduction to the Einstein equations and related PDEs.

Part 2: Positive mass theorems.

Part 3: Mass/angular momentum inequalities.

Part 1: Introduction to the Cauchy problem

We first recall the basic set up in General Relativity.

Mathematical Model: $\mathcal{S}^4$ is a smooth manifold with a Lorentz signature metric g. This means that for any point $p \in \mathcal{S}$ we can find a Lorentz basis e_0, e_1, e_2, e_3 for the tangent space so that $g_{ab} = \epsilon_a \delta_{ab}$ where $\epsilon_0 = -1$ and $\epsilon_i = 1$ for $i = 1, 2, 3$.

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$T_p\mathcal{S}$

Lightcone Structure

$v \in T_p\mathcal{S}$

$\langle v, v\rangle > 0$ Spacelike

$\langle v, v\rangle < 0$ timelike

$\langle v, v\rangle = 0$ Null

The Einstein Equations

Matter in relativity is represented by tensor fields over $\mathcal{S}$, and the spacetime metric g represents the gravitational field. The matter fields evolve from initial data via their equations of motion, and the gravitational field evolves via the Einstein equation

$$Ric(g) - \frac{1}{2} R\ g = 8\pi T$$

where Ric denotes the Ricci curvature and $R = Tr_g(Ric(g))$ is the scalar curvature.

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where Ric denotes the Ricci curvature and $R = Tr_g(Ric(g))$ is the scalar curvature.

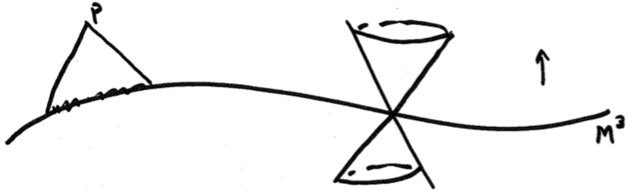
When there are no matter fields present the right hand side T is zero, and the equation reduces to

$$Ric(g) = 0.$$

These equations are called the vacuum Einstein equation.

Initial Data

The solution is determined by initial data given on a spacelike hypersurface M^3 in $\mathcal{S}$.



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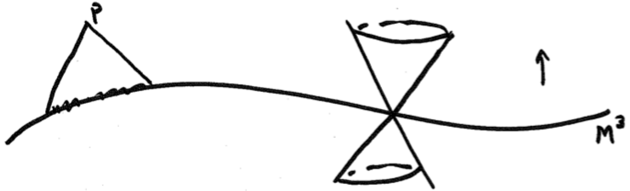

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The initial data for g are the induced (Riemannian) metric, also denoted g, and the second fundamental form p. These play the role of the initial position and velocity for the gravitational field. An initial data set is a triple (M, g, p).

The Constraint Equations

Using the Einstein equations together with the Gauss and Codazzi equations, the constraint equations may be written

$$\mu = \frac{1}{16\pi}(R_M + Tr_g(p)^2 - \|p\|^2)$$

$$J_i = \frac{1}{8\pi}\sum_{j=1}^{3} \nabla^j \pi_{ij}$$

for $i = 1, 2, 3$ where $\pi_{ij} = p_{ij} - Tr_g(p) g_{ij}$.

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for $i = 1, 2, 3$ where $\pi_{ij} = p_{ij} - Tr_g(p)g_{ij}$.

In case there is no matter present, the vacuum constraint equations become

$$R_M + Tr_g(p)^2 - \|p\|^2 = 0$$
$$\sum_{j=1}^{3} \nabla^j \pi_{ij} = 0$$

for $i = 1, 2, 3$ where R_M is the scalar curvature of M.

Energy Conditions

For spacetimes with matter, the stress-energy tensor is normally required to satisfy the **dominant energy condition** which says that the energy-momentum density 4-vector of the matter fields is non-spacelike for any observer. For an initial data set this is the inequality $\mu \geq \|J\|$.

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In the time symmetric case ($p = 0$) the dominant energy condition is equivalent to the inequality $R_M \geq 0$. In case the maximal case $Tr_g(p) = 0$ the dominant energy condition implies $R_M \geq 0$

Mean curvature in relativity

The notion of trapped surface is related to black holes in relativity and this is expressed in terms of a mean curvature inequality:

$$H(\Sigma) - Tr_{\Sigma}(p) < 0$$

means that a surface Σ is outer trapped. If the initial data contains such a surface the spacetime will become singular.

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PDEs related to the mean curvature which are important in GR:
- $H(\Sigma) = 0$, minimal surface equation, stability is key
- $H(\Sigma) - Tr_{\Sigma}(p) = 0$, MOTS equation, stability
- Inverse mean curvature flow,

$$\frac{\partial X}{\partial t} = \frac{1}{H}\nu(X(t)).$$

Asymptotic Flatness

The most natural boundary condition for the Einstein equations is the condition of asymptotic flatness. This boundary condition describes isolated systems which are the analogues of finite mass distributions in Newtonian gravity. The requirement is that the initial manifold M outside a compact set be diffeomorphic to the exterior of a ball in R^3 and that there be coordinates x in which g and p have appropriate falloff

$$g_{ij} = \delta_{ij} + O_2(|x|^{-1}), \; p_{ij} = O_1(|x|^{-2}).$$

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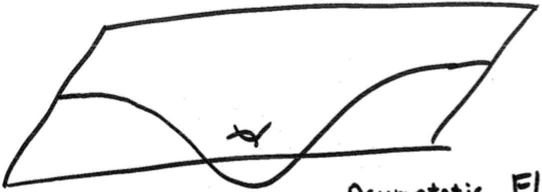

Asymptotic Flatness

Minkowski and Schwarzschild Solutions

The following are two basic examples of asymptotically flat spacetimes:

1) The Minkowski spacetime is R^{n+1} with the flat metric $g = -dx_0^2 + \sum_{i=1}^{n} dx_i^2$. It is the spacetime of special relativity.

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2) The Schwarzschild spacetime is determined by initial data with $p = 0$ and

$$g_{ij} = (1 + \frac{E}{2|x|^{n-2}})^{\frac{4}{n-2}} \delta_{ij}$$

for $|x| > 0$. It is a vacuum solution describing a static black hole with mass E. It is the analogue of the exterior field in Newtonian gravity induced by a point mass.

ADM Energy and Linear Momentum

For general asymptotically flat initial data sets there is a notion of total (ADM) energy-momentum. These quantities are computed in terms of the asymptotic behavior of g and p. For these definitions we fix asymptotically flat coordinates x and we set
$\pi = p - Tr(p)\, g$.

$$E = \tfrac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_0^j \, d\sigma_0$$

$$P_i = \tfrac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{j=1}^{n} \pi_{ij}\nu_0^j \, d\sigma_0, \qquad i = 1, 2, \ldots, n$$

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These limits exist under quite general asymptotic decay conditions. For the constant time slices in the Schwarzschild metric we have $E = m$. Generally (E, P) can be thought of as a 4-vector in the asymptotic Minkowski space, and for a more general slice in these spacetimes we have $m = \sqrt{E^2 - |P|^2}$.

Part 2: An improved positive mass theorem

We will describe the proof of the following theorem due to (EHLS) M. Eichmair, L. Huang, D. Lee, and the speaker (arXiv:1110.2087).

Theorem (Spacetime positive mass theorem)

Let $3 \leq n < 8$, and let (M, g, k) be an n-dimensional asymptotically flat initial data set satisfying the dominant energy condition. Then

$$E \geq |P|,$$

where (E, P) is the ADM energy-momentum vector of (M, g, k).

Previous Results

Our theorem is an improvement of earlier results.

• $R \geq 0$ implies $E \geq 0$ by S & Yau for $3 \leq n < 8$. This includes the maximal (and Riemannian) case.

• Dominant energy condition implies $E \geq 0$. Done by S & Yau for n=3, and the method extended recently by Eichmair for $3 < n < 8$.

• For spin manifolds of any dimension $E \geq |P|$ follows from argument of E. Witten.

• For $n = 3$, the statement $R \geq 0$ implies $E \geq 0$ also follows from the inverse mean curvature flow by an argument proposed by R. Geroch and made rigorous by G. Huisken & T. Ilmanen. The argument also gives more quantative statements such as the Penrose inequality in case M has a compact connected outermost minimal boundary (black hole).

The Stability Condition

The stability condition for minimal hypersurfaces expresses the condition that the second variation of volume is non-negative for variations of Σ. It may be written

$$\int_\Sigma [\|\nabla\varphi\|^2 - (\|A\|^2 + Ric(\nu,\nu))\varphi^2]\ dv \geq 0$$

for all functions φ of compact support. This expresses the condition that the second variation of volume is nonnegative for a variation in the direction $\varphi \cdot \nu$ where ν is a unit normal vector to Σ.

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Using the Gauss equation the stability condition may be written

$$\int_\Sigma [\|\nabla\varphi\|^2 - \frac{1}{2}(R^M - R^\Sigma + \|A\|^2)\varphi^2]\ dv \geq 0.$$

Stable MOTS

For the MOTS equation

$$H(\Sigma) - Tr_{\Sigma}(p) = 0$$

there is a notion of stability which is essentially the condition that Σ lies in a local foliation Σ_t with $\Sigma_0 = \Sigma$ so that

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By an interesting calculation this implies an eigenvalue condition of the form

$$\int_{\Sigma} [\|\nabla\varphi\|^2 - \frac{1}{2}((\mu - |J|) - R^{\Sigma} + \|A - p\|^2)\varphi^2] \; dv \geq 0$$

for φ with compact support.

Finding stable MOTS; the Jang equation

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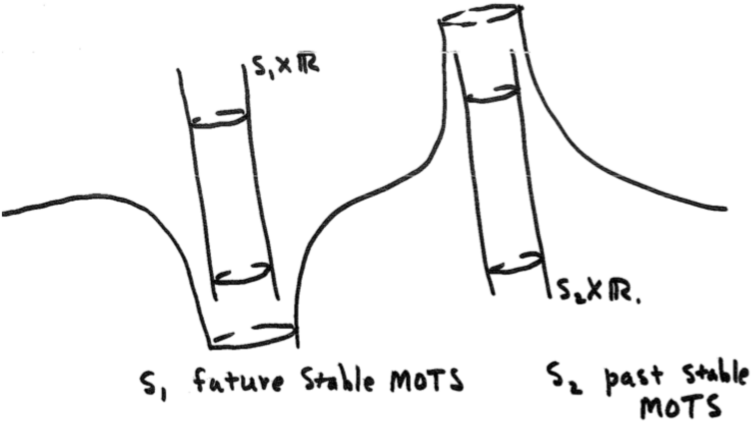
The speaker and Yau reduced the spacetime positive energy theorem to the Riemannian case by constructing a graphical solution of this equation on $M \times \mathbb{R}$; this is the Jang equation

$$div\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) = \sum_{i,j=1}^{3}\left(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}\right)p_{ij}.$$

The left hand side is the mean curvature of the graph of f and the right hand side is the trace of (the extended) p along the graph.

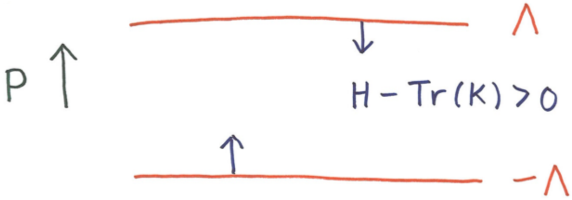
Blow-up on stable MOTS

The only known way to construct stable MOTS is by constructing solutions of the Jang equation which blow up on an interface which is then a stable MOTS.



EHLS Proof of the Spacetime Positive Mass Theorem

We assume we are in special asymptotics and we show that if $E < |P|$ then we have the picture (reminiscent of the Riemannian case)



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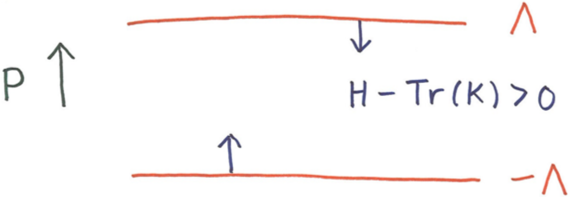

This is based on the calculation in special asymptotics on the hyperplanes $x_n = \Lambda$ where we have chosen coordinates for which P points in the positive x_n direction. The proof involves a study of asymptotically planar stable MOTS.

The use of stability

In three dimensions it is possible to use the stability condition together with the Gauss-Bonnet theorem to show that there can be no asymptotically planar stable MOTS in an initial data set satisfying the dominant energy condition.

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For $n \geq 4$ an additional difficulty appears since we need some variations which do not have compact support; that is, we need to construct a strongly stable MOTS in the sense that we can allow variations which are vertical translations near infinity. This was accomplished by an extra minimization in the Riemannian case. An interesting and subtle feature of the argument is that we are able to accomplish this even though the equation is not variational.

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If we have $E < |P|$, we construct a strongly stable MOTS Σ and use the strong stability condition to find an asymptotically flat metric on Σ with $R = 0$ and $E < 0$. This contradicts the Riemannian postive energy theorem in dimension $n - 1$.

Key technical ingredients

- A perturbation theorem to simplify the asymptotics keeping the constraint equations valid. This is an example of a *density theorem* for the constraint equations.

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- Constructing Barriers and proving existence of stable MOTS which are asymptotically planar.

- For $n \geq 4$ we need to choose the height correctly in order that our stable MOTS is strongly stable in the sense that we can allow a variation which is a vertical translation near infinity.

Part 3: Mass/angular momentum inequalities: the Kerr Solutions

There is a family of solutions depending on two parameters m and α where m is mass and $|J| = \alpha^2$ is the angular momentum. These reduce to the Schwarzschild solution when $\alpha = 0$ and they represent stationary rotating black hole solutions. In order to represent black hole solutions it is necessary that the **Kerr constraint** $\sqrt{|J|} \leq m$ hold.

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The Kerr metric is given in coordinates (t, r, ϕ, θ) by

$$g = -dt^2 + \frac{2mr}{\rho^2}(\alpha \sin^2\phi d\theta - dt)^2 + \rho^2\left(\frac{dr^2}{\Delta} + d\phi^2\right) + (r^2 + \alpha^2)\sin^2\phi d\theta^2$$

where $\rho^2 = r^2 + \alpha^2\cos^2\phi$ and $\Delta = r^2 - 2mr + \alpha^2$. In order for the metric to be nonsingular in these coordinates we require $r > m + \sqrt{m^2 - \alpha^2}$, the largest root of $\Delta(r)$.

ADM angular momentum

Recall the definition of the energy and linear momentum.

$$
\begin{aligned}
E &= \frac{1}{16\pi} \lim_{R\to\infty} \oint_{|x|=R} \sum_{i,j} \left(g_{ij,i} - g_{ii,j}\right) \nu^j d\sigma_g \\
P_i &= \frac{1}{8\pi} \lim_{R\to\infty} \oint_{|x|=R} \sum_{j} \pi_{ij} \nu^j d\sigma_g, \;\; i = 1, 2, 3
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\end{aligned}
$$

Under appropriate asymptotic conditions the angular momentum can be defined in a similar way.

$$
J = \frac{1}{8\pi}\lim_{R\to\infty}\oint_{|x|=R}\sum_{j,k}\pi_{jk}Y^j\nu^k d\sigma_g
$$

where $Y = \frac{\partial}{\partial x^i} \times \vec{x}$ (cross product) is the oriented rotation vector field around the x^i-axis.

Some background

Since the Kerr solutions are the expected final states of gravitational collapse, it is important to be able to characterize them in a robust way. For example, does the restriction $\sqrt{|J|} \leq m$ hold for a natural class of dynamical solutions?

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- It was shown by X. Zhang in 1999 that the Kerr constraint holds for general data which satisfy an energy condition which is more stringent than the dominant energy condition.
- It was shown by S. Dain in 2008 that a large class of (non-stationary) axisymmetric vacuum black hole solutions do satisfy the Kerr constraint. The work was extended by P. Chruściel, J. Costa, Y. Y Li, L. Nguyen, G. Weinstein.
- We will describe recent joint work with Xin Zhou (arXiv:1209.0019) which proves all of the known results in a stronger form by a simpler argument.

Axisymmetric data and maps to the hyperbolic plane

Given a maximal ($Tr(p) = 0$) vacuum initial data (M, g, p) with a spacelike Killing vector field η having closed orbits, we may associate a map $(X, Y) : M \to H^2$ where

$$H^2 = \{(X, Y) : \ X > 0\}, \ X^{-2}(dX^2 + dY^2)$$

where $X = \|\eta\|^2$ and $\frac{1}{2} dY = *(i_\eta h \wedge \eta_\#)$. The 1-form defining Y is closed because of the vacuum constraint equations and the maximal condition.

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where $X = \|\eta\|^2$ and $\frac{1}{2}dY = *(i_\eta h \wedge \eta_\#)$. The 1-form defining Y is closed because of the vacuum constraint equations and the maximal condition.

The map (X, Y) is harmonic if and only if (M, g, p) defines a stationary solution of the vacuum Einstein equations.

Extreme Kerr as a Harmonic Mapping

There is a classical description (due to B. Carter) of the Kerr solution as a harmonic mapping u_0 from $\mathbb{R}^3$ into the hyperbolic plane $H^2 = \{(X, Y) : \; X > 0\}$ with metric $X^{-2}(dX^2 + dY^2)$. Explicitly we have the extremal Kerr solution ($m = \sqrt{|J|}$) corresponding to $u_0 = (X_0, Y_0)$ where

$$X_0 = \big(\tilde{r}^2 + |J| + \frac{2|J|^{3/2}\tilde{r}\sin^2\theta}{\Sigma}\big)\sin^2\theta$$

$$Y_0 = 2J(\cos^3\theta - 3\cos\theta) - \frac{2J^2\cos\theta\sin^4\theta}{\Sigma}$$

and

$$\tilde{r} = r + \sqrt{|J|}, \quad \Sigma = \tilde{r}^2 + |J|\cos^2\theta, \tag{1}$$

where r, θ, φ are spherical coordinates in $\mathbb{R}^3$ and J is the angular momentum of the Kerr solution.

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where r, θ, φ are spherical coordinates in $\mathbb{R}^3$ and J is the angular momentum of the Kerr solution.

Note that u_0 is singular at the origin (if $J \neq 0$), and X_0 vanishes along the z axis Γ like ρ^2 where $\rho = r\sin\theta$.

The Brill Initial Data Sets

The axisymmetric, maximal solutions of the vacuum constraint equations also have descriptions as (non-harmonic) maps $u = (X, Y) : \mathbb{R}^3 \to H^2$. There is a large class of black hole solutions with angular momentum J which are asymptotic to the extreme Kerr solution. These give rise to maps u which are asymptotic to u_0 near Γ. The maps u and u_0 have infinite energy, but there is a natural renormalized energy $\mathcal{M}(u)$. This is defined by writing $x = \log X - g$ where $g = 2\log\rho$. We then define

$$\mathcal{M}(u) = \int_{R^3} [|\nabla x|^2 + X^{-2}|\nabla Y|^2] \, d\mu.$$

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$$\mathcal{M}(u) = \int_{R^3} [|\nabla x|^2 + X^{-2}|\nabla Y|^2]\, d\mu.$$

We have that $\mathcal{M}(u_0)$ is finite, and $\mathcal{M}(u)$ is also finite provided that $x - x_0 \in H^1(\mathbb{R}^3)$, $(x - x_0)_- \in L^\infty(\mathbb{R}^3)$, and $Y - Y_0 \in H^1_{X_0}(\mathbb{R}^3 \setminus \Gamma)$ defined as the completion of $C_c^\infty(\mathbb{R}^3 \setminus \Gamma)$ with respect to the norm

$$\|f\|^2_{X_0} = \int_{R^3} X_0^{-2}|\nabla f|^2\, d\mu.$$

Properties of the Renormalized Energy

If we take a domain Ω which is compactly contained in $\mathbb{R}^3 \setminus \Gamma$, then we have that $E_\Omega(u)$ and $\mathcal{M}_\Omega(u)$ differ by a boundary term. (This uses the fact that $\log \rho$ is a harmonic function in $\mathbb{R}^3$.) It follows that the two functionals have the same stationary maps; that is, the harmonic maps.

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It turns out that the renormalized energy is related to the mass. First we have $\mathcal{M}(u_0) = m_0 = \sqrt{|J|}$, and we have the following inequality.

Theorem (D. Brill, S. Dain) For the data described above we have $m \geq \mathcal{M}(u)$

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If we take a domain Ω which is compactly contained in $\mathbb{R}^3 \setminus \Gamma$, then we have that $E_\Omega(u)$ and $\mathcal{M}_\Omega(u)$ differ by a boundary term. (This uses the fact that $\log \rho$ is a harmonic function in $\mathbb{R}^3$.) It follows that the two functionals have the same stationary maps; that is, the harmonic maps.

It turns out that the renormalized energy is related to the mass. First we have $\mathcal{M}(u_0) = m_0 = \sqrt{|J|}$, and we have the following inequality.

Theorem (D. Brill, S. Dain) For the data described above we have $m \geq \mathcal{M}(u)$

Thus the inequality $m \geq \sqrt{|J|}$ follows if we can show that $\mathcal{M}(u) \geq \mathcal{M}(u_0)$ for the maps which arise from our data. Thus the mass/angular momentum inequality follows from the condition that the harmonic map u_0 minimizes its renormalized energy in an appropriate class of competing maps.

Convexity of the Renormalized Energy

There is a convexity result for the renormalized energy which generalizes convexity for maps from compact manifolds to manifolds of non-positive curvature.

Theorem (R. S. & Xin Zhou) Assume that $u_0 = (X_0, Y_0)$ is the extremal Kerr map, and that $u_1 = (X_1, Y_1)$ is another map with $x_1 - x_0 \in H^1(\mathbb{R}^3)$, $(x_1 - x_0)_- \in L^\infty(\mathbb{R}^3)$, and $Y_1 - Y_0 \in H^1_{X_0}(\mathbb{R}^3 \setminus \Gamma)$. If u_t is the geodesic path of maps from u_0 to u_1, then we have

$$\frac{d^2}{dt^2}\mathcal{M}(u_t) \geq 2 \int_{R^3} \|\nabla d(u_0, u_1)\|^2.$$

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The proof uses the same basic calculation as in the compact case, but requires a delicate argument to handle the singularity. The explicit nature of the singularity and the specifics of geodesics in H^2 is used.

A Quantitative Version of the Mass/Angular Momentum Inequality

We may now apply the Sobolev inequality to prove the following.

Theorem (R. S. & Xin Zhou) Assume that $u_0 = (X_0, Y_0)$ is the extremal Kerr map, and that $u_1 = (X_1, Y_1)$ is another map with $x_1 - x_0 \in H^1(\mathbb{R}^3)$, $(x_1 - x_0)_- \in L^\infty(\mathbb{R}^3)$, and $Y_1 - Y_0 \in H^1_{X_0}(\mathbb{R}^3 \setminus \Gamma)$. We have the bound

$$\mathcal{M}(u_1) - \mathcal{M}(u_0) \geq C \| d(u_0, u_1) \|^2_{L^6}$$

where $C = \frac{3}{4}(2\pi^2)^{2/3}$.

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where $C = \frac{3}{4}(2\pi^2)^{2/3}$.

Corollary If u_1 corresponds to a Brill data set with mass m and angular momentum J, then it follows that

$$m - \sqrt{|J|} \geq C\|d(u_0, u_1)\|^2_6.$$

Extensions of the results

P. Chruściel and his collaborators (Y. Li and G. Weinstein) gave a strong generalization by allowing asymptotic behavior which corresponds to general (non-extreme) Kerr data. Chruściel and J. Costa also extended the bound to the Einstein/Maxwell case. We also treat these cases and improve the results and simplify the proofs by the convexity method.

Non-axisymmetric solutions

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Theorem (L. H. Huang, R. S., M. Wang) Let (g, p) be a nontrivial vacuum initial data set satisfying appropriate asymptotic conditions. Given any constant vectors $\vec{\alpha}_0, \vec{\gamma}_0 \in \mathbb{R}^3$, there exists a vacuum initial data set $(\bar{g}, \bar{p})$ which is a perturbation of (g, p) so that

$$\overline{E} = E, \quad \overline{P} = P,$$

and

$$\overline{J} = J + \vec{\alpha}_0, \quad \overline{C} = C + \vec{\gamma}_0.$$

Open questions

• Prove the inequality without the maximal assumption $Tr(p) = 0$. X. Zhou posted a paper where he proves the inequality assuming smallness of $Tr(p)$. Can the formal spacetime argument of finding a maximal slice be made rigorous?

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- Does the mass/angular momentum inequality hold for other natural classes of data? How about almost axisymmetric?

- Can the inequality be proven for data with a black hole boundary? The current results require complete data with multiple ends.