The (Variable Coefficient) Thin Obstacle Problem: A Carleman Approach

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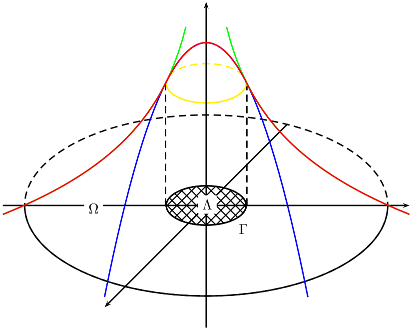
The (Classical) Obstacle Problem

Minimize

$$J(w) := \int_{B_1} |\nabla w|^2 dx \text{ in } \mathcal{K} := \{w \in H^1_g(B_1^+) |\ w \geq \phi \text{ in } B_1\}.$$



Questions:

- Existence & uniqueness,
- Regularity of w,
- Regularity of Γ.

Literature:

- Friedman: Variational Principles and Free-Boundary Problems,
- Caffarelli: The Obstacle Problem Revisited.

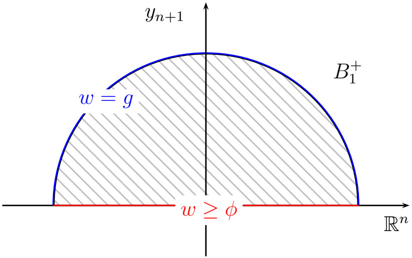
The Thin Obstacle Problem

Minimize

$$J(w) := \int_{B_1^+} |\nabla w|^2 dx,$$

in the convex set

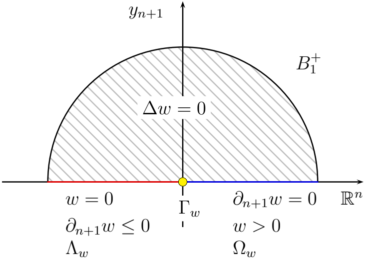
$$\mathcal{K} := \{w \in H^1_g(B_1^+) |\ w \geq \phi \text{ on } B_1' = B_1^+ \cap \{y_{n+1} = 0\}\}.$$



Applications:

- semipermeable membranes (osmosis),
- optimal pricing (American options),
- Signorini problem in elasticity.

$$\text{Minimize} \int_{B_1^+} |\nabla w|^2 dx \text{ in } \mathcal{K} := \{w \in H^1_g(B_1^+) |\ w \geq 0 \text{ on } B_1'\}.$$



$$\boxed{\begin{aligned} \Delta w &= 0 \text{ in } B_1^+, \\ w &\geq 0 \text{ on } B_1', \\ \partial_{n+1} w &\leq 0 \text{ on } B_1', \\ w\partial_{n+1} w &= 0 \text{ on } B_1'. \end{aligned}}$$

Interesting quantities:

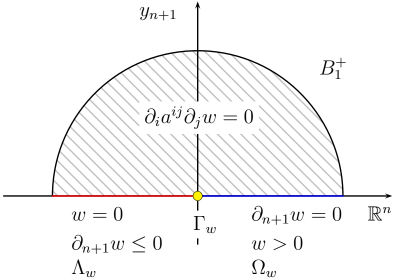
- positivity set: $\Omega_w := \{x \in \mathbb{R}^n \times \{0\} |\ w(x) > 0\}$,
- free boundary: $\Gamma_w := \partial_{B_1'} \Omega_w$,
- contact set: $\Lambda_w := B_1' \setminus \Omega_w$.

The Thin Obstacle Problem

$$\boxed{\begin{aligned} \partial_i a^{ij}\partial_j w &= 0 \text{ in } B_1^+,\\ w\geq 0,\ a^{n+1,j}\partial_j w\leq 0,\ w(a^{n+1,j}\partial_j w) &= 0 \text{ on } B_1'. \end{aligned}}$$

Aim:

- Regularity of w under weak assumptions on a^{ij}.



Assumptions: Metric

- ellipticity,
- regularity: $a^{ij}\in W^{1,p}(B_1^+)$, $p\in(n+1,\infty]$,
- normalization: $a^{ij}(0)=\delta^{ij}$; $a^{n+1,j}(x',0)=0,\ j\in\{1,...,n\}$.

Assumptions: Solution

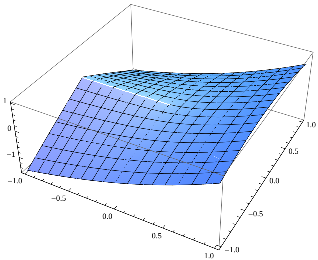
- $\|w\|_{L^2(B_1^+)}=1$,
- $w\in C^{1,\alpha}(B_{1/2}^+)$ for some $\alpha>0$.

The Thin Obstacle Problem

Proposition

For $a^{ij} = \delta^{ij}$:

- $w \in C^{1,\frac{1}{2}}(B^+_{\frac{1}{2}})$,
- $\Gamma_w = \Gamma_{3/2}(w) \cup \bigcup\limits_{\kappa \geq 2} \Gamma_w$,
- $\Gamma_{3/2}(w) \in C^{1,\alpha}$ *for some* $\alpha \in (0,1)$.



Literature:

- Lewy (1972), Richardson (1978),
- Caffarelli (1979), Kinderlehrer (1981), Uraltseva (1987),
- Caffarelli, Athanasopoulos, Salsa, Silvestre (2006-2008),
- Garofalo, Petrosyan, Smit Vega Garcia (2014, 2015),
- Focardi, Spadaro (2015).

The Thin Obstacle Problem: The Classical Approach

- Frequency function:

$$r \mapsto N(r) = N(r,w) := \frac{r\int_{B_r} |\nabla w|^2}{\int_{\partial B_r} w^2}$$

  is monotone non-decreasing.
- If $N(r,w) = \kappa$ for all $r \in (0,R)$, then w is κ homogeneous in B_R.
- Link homogeneous, global solutions to general solutions via blow-up: If

$$w_r(x) := \frac{w(rx)}{\left(\frac{1}{r^{n-1}}\int\limits_{\partial B_r} w^2\right)^{1/2}},$$

  then $N(\rho, w_r) = N(r\rho, w)$.

The Thin Obstacle Problem: The Classical Approach

- Convergence: $w_r \to w_0$ in $C^1(B_1)$, where w_0 is $N(0+, w)$ homogeneous.

- Classification of lowest possible homogeneity: $N(0+, w) \geq \frac{3}{2}$,

$$w_{3/2}(x) = C_n \,\mathrm{Re}(x_n + i x_{n+1})^{3/2}.$$

- Growth estimates:

$$\sup_{B_r} |w| \leq C r^{3/2} \text{ for } 0 < r < 1/2.$$

Main Results

Proposition (Almost Optimal Regularity)

Let $w : B_1^+ \to \mathbb{R}$ be a solution of the variable coefficient thin obstacle problem. Let $\gamma = 1 - \frac{n+1}{p}$. Then,

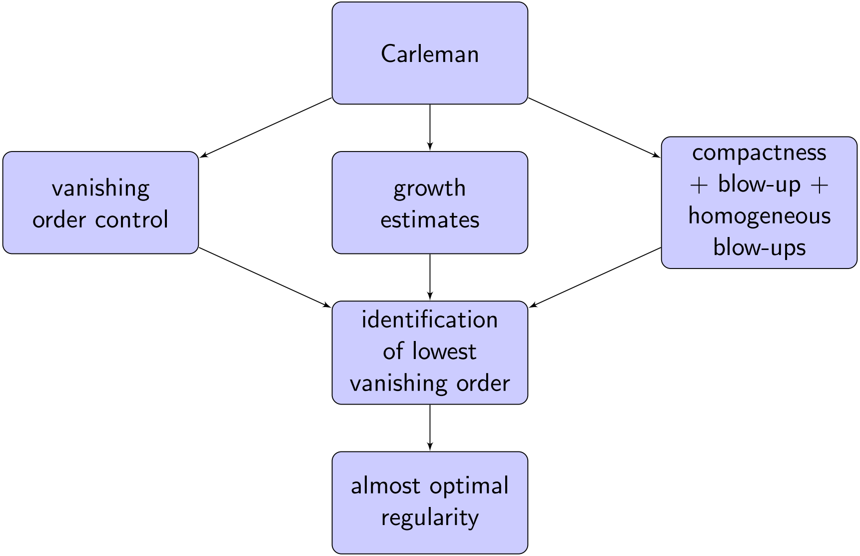
$$|\nabla w(x) - \nabla w(y)| \leq \begin{cases} C(\gamma)|x-y|^{\gamma} & \text{for } p \in (n+1, 2(n+1)), \\ C|x-y|^{1/2}\ln(|x-y|)^2 & \text{for } p \geq 2(n+1), \end{cases}$$

for all $x, y \in B_1^+$.

Additional Results:

- growth estimates,
- compactness of blow-up limits,
- homogeneous blow-up limits (without Friedland-Hayman),
- lower semi-continuity of the vanishing order.

Outline of the Strategy



The Carleman Estimate

Proposition (Carleman)

Let $0<\rho<r<1$. *Assume that* $w \in H^1(B_1^+)$ *with* $\operatorname{supp}(w) \subset \overline{A_{\rho,r}^+}$ *solves*

$$\begin{aligned}\partial_i a^{ij}\partial_j w &= f \text{ in } A_{\rho,r}^+,\\ w\geq 0,\ \partial_{n+1}w\leq 0,\ w\partial_{n+1}w &= 0 \text{ on } A'_{\rho,r},\end{aligned}$$

where $f: A_{\rho,r}^+ \rightarrow \mathbb{R}$ *is an in* $\overline{A_{\rho,r}^+}$ *compactly supported* $L^2(A_{\rho,r}^+)$ *function. Then for* $\gamma = 1-\frac{n+1}{p}$ *and any* $\tau\geq 1$ *we have*

$$\begin{aligned}&\tau^{\frac{3}{2}}\left\|e^{\tau\psi}|x|^{-1}(1+\ln(|x|)^2)^{-\frac{1}{2}}w\right\|_{L^2(A_{\rho,r}^+)} + \tau^{\frac{1}{2}}\left\|e^{\tau\psi}(1+\ln(|x|)^2)^{-\frac{1}{2}}\nabla w\right\|_{L^2(A_{\rho,r}^+)}\\ &\leq C(n,\phi)\left(\tau^2\left\|\nabla a^{ij}\right\|_{L^p(A_{\rho,r}^+)}\left\|e^{\tau\psi}|x|^{\gamma-1}w\right\|_{L^2(A_{\rho,r}^+)} + \left\|e^{\tau\psi}|x|f\right\|_{L^2(A_{\rho,r}^+)}\right).\end{aligned}$$

- low regularity of coefficients,
- boundary contributions.

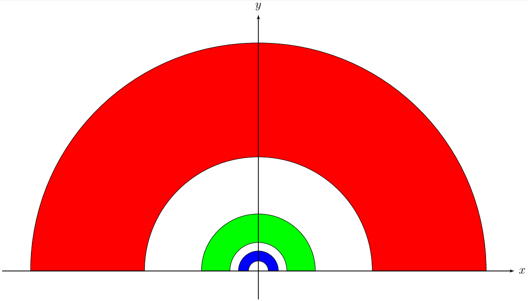
Consequences of the Carleman estimate

Corollary (Three spheres inequality)

Let $1 \leq \tau \leq \tau_0 < \infty$ *and* $0 < r_1 < r_2 < r_3 \leq R_0(\tau_0)$, *then*

$$\tau^{\frac{3}{2}}(1+|\ln(r_2)|)^{-1} e^{\tau\tilde{\psi}(\ln(r_2))} r_2^{-1} \|w\|_{L^2(A^+_{r_2,2r_2}(x_0))}$$
$$\leq C\left(e^{\tau\tilde{\psi}(\ln(r_1))} r_1^{-1} \|w\|_{L^2(A^+_{r_1,2r_1}(x_0))} + e^{\tau\tilde{\psi}(\ln(r_3))} r_3^{-1} \|w\|_{L^2(A^+_{r_3,2r_3}(x_0))}\right).$$

- "weak *logarithmic convexity*" of the L^2 norm



Vanishing Order and Asymptotic Growth

Definition (Vanishing Order)

For any $u \in L^2(B_\delta^+(x_0))$, $\delta > 0$, we define the vanishing order of u at x_0 as

$$\kappa_{x_0} := \limsup_{r \to 0} \frac{\ln\left(r^{-(n+1)} \int_{A^+_{r/2,r}(x_0)} u^2\right)^{1/2}}{\ln(r)} \in [-\infty, \infty].$$

Consequences of the Carleman inequality:

- $\limsup\limits_{r\to 0} \frac{\ln\left(r^{-(n+1)} \int_{A^+_{r/2,r}(x_0)} w^2\right)^{1/2}}{\ln(r)} = \liminf\limits_{r\to 0} \frac{\ln\left(r^{-(n+1)} \int_{A^+_{r/2,r}(x_0)} w^2\right)^{1/2}}{\ln(r)}$,
- Uraltseva: $\kappa_{x_0} \geq 1 + \alpha$.

Growth Estimates

Lemma (Growth estimates)

For any $\epsilon > 0$ *there exists a radius* $r_\epsilon = r_\epsilon(w) > 0$ *such that*

$$r^{\kappa_{x_0}+\frac{n+1}{2}+\epsilon} \leq \|w\|_{L^2(A^+_{r/2,r}(x_0))} \leq r^{\kappa_{x_0}+\frac{n+1}{2}-\epsilon} \text{ for all } 0 < r \leq r_\epsilon.$$

Proposition (Uniform upper growth bounds)

Given a finite constant $\bar{\kappa} > 0$ *there exists a constant* $C = C(\kappa, n, p, \left\|\nabla a^{ij}\right\|_{L^p(B_1^+)})$ *such that for all* $x_0 \in \Gamma_w \cap B'_{1/2}$

$$\sup_{B_r^+(x_0)} |w| \leq C r^{\min\{\kappa_{x_0}, \bar{\kappa}\}} |\ln(r)|^2 \text{ for all } 0 < r \leq R_0.$$

Compactness and Blow-up

Proposition (Blow-up limit)

Let w be a (non-trivial) solution of the thin obstacle problem and let $0 \in \Gamma_w$ with $\kappa_0 < \infty$. Then (along subsequences):

$$w_\sigma(x) := \frac{w(\sigma x)}{\sigma^{-\frac{n+1}{2}} \|w\|_{L^2(B_\sigma^+(0))}} \to w_0(x) \text{ in } L^2(B_1^+) \text{ as } \sigma \to 0.$$

Moreover, w_0 solves the thin obstacle problem with constant coefficients:

$$\Delta w_0 = 0 \text{ in } B_1^+,$$
$$w_0 \geq 0,\ -\partial_{n+1} w_0 \geq 0,\ w_0(\partial_{n+1} w_0) = 0 \text{ on } B_1'.$$

Furthermore, $\|w_0\|_{L^2(B_1^+)} = 1$; in particular, it is not the trivial function.

Compactness via doubling: $\|w\|_{L^2(B_r^+)} \leq C \|w\|_{L^2(B_{r/2}^+)}$.

Homogeneous Blow-Up Limits

Lemma (Almost homogeneity)

Let $w : B_1^+ \rightarrow \mathbb{R}$ *be a solution of the variable coefficient thin obstacle problem. Let* $\kappa_0 > 0$ *be the vanishing order at* 0*. There exists a sequence of radii* $r_j \rightarrow 0$ *and parameters* $\epsilon_j \rightarrow 0$ *such that*

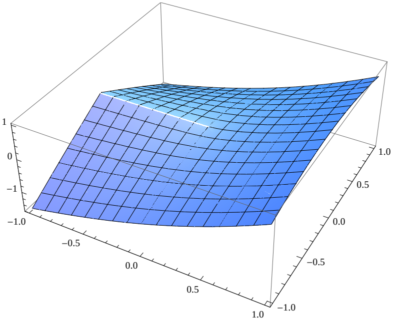
$$\left\| x \cdot \nabla w_{r_j} - \kappa_0 w_{r_j} \right\|_{L^2(A_{1/2,1})} \leq \epsilon_j .$$

Consequence: Existence of homogeneous blow-up limits

There exists a sequence of radii $\{r_j\}_{j\in\mathbb{N}}$ with $r_j \rightarrow 0$ such that

- $w_{r_j} \rightarrow w_0$ as $j \rightarrow 0$,
- w_0 is a homogeneous solution of the constant coefficient equation with homogeneity κ_0.

Homogeneous Blow-up Limits



Proposition (Global homogeneous solutions, PSU)

Let w_0 be a homogeneous global solution of the thin obstacle problem with homogeneity $\kappa \in (1,2)$. Then $\kappa = 3/2$ *and*

$$w_{3/2}(x) = C_n \operatorname{Re}(x_n + i x_{n+1})^{3/2}.$$

up to multiplication by a constant and a rotation in $\mathbb{R}^n$.

Corollary

Let w be a solution of the variable coefficient thin obstacle problem and assume that $x_0 \in \Gamma_w \cap B_1'$. Then either $\kappa_{x_0} = \frac{3}{2}$ or $\kappa_{x_0} \geq 2$.

Almost Optimal Regularity – Proof

- $\kappa_{x_0} \geq \frac{3}{2}$ for all $x_0 \in \Gamma_w \cap B_1'$,
- growth estimate:

$$|w(x)| \leq C \operatorname{dist}(x, \Gamma_w)^{3/2} |\ln(\operatorname{dist}(x, \Gamma_w))|^2$$

  for all $0 < |x| \leq R_0$,
- (even or odd) reflection and interior elliptic estimates:

$$|\nabla u(x) - \nabla u(y)| \leq C|x-y|^{\frac{1}{2}} \ln(|x-y|)^2.$$

Outlook

- Separation of the free boundary into the regular free boundary and the rest:

$$\Gamma_w = \Gamma_{3/2}(w) \cup \bigcup_{\kappa \geq 2} \Gamma_\kappa(w).$$

- Analysis of the regular free boundary: $C^{1,\alpha}$ regularity for some $\alpha \in (0,1)$.

- Optimal regularity estimates for w (free boundary regularity + Carleman).

- Identification of asymptotic expansion of solutions at the regular free boundary.