A dezornification of the proof of the existence of a maximal Cauchy development for the Einstein equations

Jan Sbierski

University of Cambridge

23.03.2015

The main theorem

Theorem (Choquet-Bruhat and Geroch '69)

Given smooth initial data for the (vacuum) Einstein equations there exists a globally hyperbolic development $\tilde{M}$ that is an extension of any other globally hyperbolic development of the same initial data. The globally hyperbolic development $\tilde{M}$ is unique up to isometry and is called the maximal globally hyperbolic development of the given initial data.

The original proof of this theorem relied crucially on the axiom of choice in the form of Zorn's lemma. However, the theorem can also be proven without assuming the axiom of choice.

The main theorem

Theorem (Choquet-Bruhat and Geroch '69)

Given smooth initial data for the (vacuum) Einstein equations there exists a globally hyperbolic development $\tilde{M}$ that is an extension of any other globally hyperbolic development of the same initial data. The globally hyperbolic development $\tilde{M}$ is unique up to isometry and is called the maximal globally hyperbolic development of the given initial data.

The original proof of this theorem relied crucially on the axiom of choice in the form of Zorn's lemma. However, the theorem can also be proven without assuming the axiom of choice.

Proving the analogous statement for a quasilinear wave equation

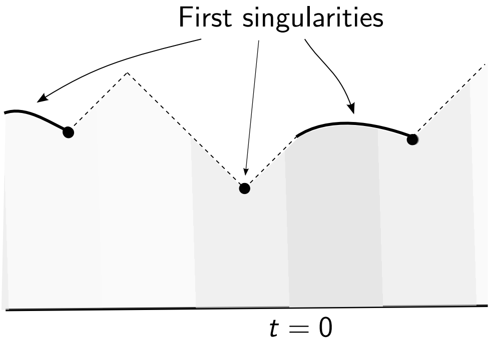
Proving the existence of a unique maximal globally hyperbolic development for a quasilinear wave equation

Consider a quasilinear wave equation

$$g^{\mu\nu}(u, \partial u)\partial_\mu \partial_\nu u = F(u, \partial u) \; ,$$

where $u : \mathbb{R}^{3+1} \supseteq \mathcal{D} \to \mathbb{R}$, and g and F are such that the initial value problem for the quasilinear wave equation is locally well-posed.

Proving the existence of a unique maximal globally hyperbolic development for a quasilinear wave equation



$u : U \to \mathbb{R}$ is a GHD $:\iff$ u is a solution, attains the initial data, and U is globally hyperbolic (wrt $g(u, \partial u)$) with Cauchy hypersurface $\{t = 0\}$.

Step 1: Global uniqueness

Prescribe initial data on $\{t = 0\}$. Let $u_1 : U_1 \to \mathbb{R}$ and $u_2 : U_2 \to \mathbb{R}$ be two GHDs. We want to show that they agree on $U_1 \cap U_2$.

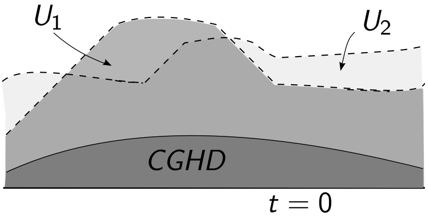
Step 1: Global uniqueness

1. By local uniqueness there exists a common globally hyperbolic development (CGHD).

2. Take union of all CGHDs, obtain MCGHD.

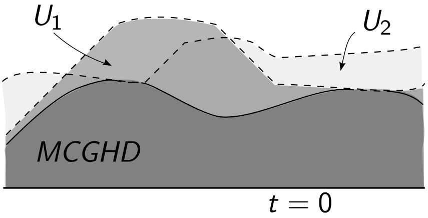
3. Assume MCGHD $\neq U_1 \cap U_2$. Find spacelike part of boundary and choose spacelike slice through such a boundary point. By continuity, both solutions agree on this slice.

4. Apply local uniqueness to this initial value problem - obtain bigger CGHD. Contradiction.



Step 1: Global uniqueness

1. By local uniqueness there exists a common globally hyperbolic development (CGHD).
2. Take union of all CGHDs, obtain MCGHD.
3. Assume MCGHD $\neq U_1 \cap U_2$. Find spacelike part of boundary and choose spacelike slice through such a boundary point. By continuity, both solutions agree on this slice.
4. Apply local uniqueness to this initial value problem - obtain bigger CGHD. Contradiction.



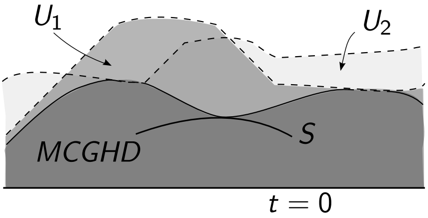
Step 1: Global uniqueness

1. By local uniqueness there exists a common globally hyperbolic development (CGHD).

2. Take union of all CGHDs, obtain MCGHD.

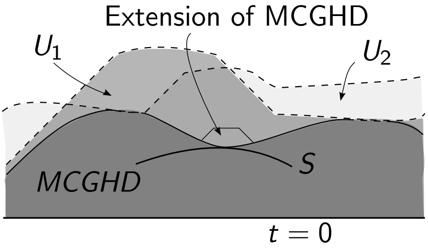
3. Assume MCGHD $\neq U_1 \cap U_2$. Find spacelike part of boundary and choose spacelike slice through such a boundary point. By continuity, both solutions agree on this slice.

4. Apply local uniqueness to this initial value problem - obtain bigger CGHD. Contradiction.



Step 1: Global uniqueness

1. By local uniqueness there exists a common globally hyperbolic development (CGHD).
2. Take union of all CGHDs, obtain MCGHD.
3. Assume MCGHD $\neq U_1 \cap U_2$. Find spacelike part of boundary and choose spacelike slice through such a boundary point. By continuity, both solutions agree on this slice.
4. Apply local uniqueness to this initial value problem - obtain bigger CGHD. Contradiction.



Step 2: Existence of MGHD

Consider the set $\{(U_\alpha, u_\alpha) \,|\, \alpha \in A\}$ of all GHDs of given initial data.
Set $U := \bigcup_{\alpha \in A} U_\alpha$ and define $u : U \to \mathbb{R}$ by

$$u(x) := u_\alpha(x) \text{ for } x \in U_\alpha.$$

This is well-defined, since global uniqueness holds.

The initial value problem in general relativity

The initial value problem in general relativity

Consider the vacuum Einstein equations

$$Ric(g) = 0$$

for a smooth $3+1$-dimensional Lorentzian manifold (M, g).

Initial data $(\overline{M}, \bar{g}, \bar{k})$ for the vacuum Einstein equations consists of a 3-dimensional Riemannian manifold $(\overline{M}, \bar{g})$ together with a symmetric 2-covariant tensor field $\bar{k}$ on $\overline{M}$ that satisfy certain *constraint equations*.

The initial value problem in general relativity

Consider the vacuum Einstein equations

$$Ric(g) = 0$$

for a smooth $3+1$-dimensional Lorentzian manifold (M, g).

Initial data $(\overline{M}, \bar{g}, \bar{k})$ for the vacuum Einstein equations consists of a 3-dimensional Riemannian manifold $(\overline{M}, \bar{g})$ together with a symmetric 2-covariant tensor field $\bar{k}$ on $\overline{M}$ that satisfy certain *constraint equations*.

The initial value problem in general relativity: definitions I

Definition

A *globally hyperbolic development* (GHD) (M, g, ι) of initial data $(\overline{M}, \bar{g}, \bar{k})$ is a time oriented, globally hyperbolic Lorentzian manifold (M, g) that satisfies the vacuum Einstein equations, together with an embedding $\iota : \overline{M} \to M$ such that

1. $\iota^*(g) = \bar{g}$
2. $\iota^*(k) = \bar{k}$, where k denotes the second fundamental form of $\iota(\overline{M})$ in M
3. $\iota(\overline{M})$ is a Cauchy surface in (M, g).

The initial value problem in general relativity: definitions II

Definition

Given two GHDs (M, g, ι) and (M', g', ι') of the same initial data, we say that (M', g', ι') is an *extension* of (M, g, ι) iff there exists a time orientation preserving isometric embedding $\psi : M \to M'$ that preserves the initial data, i.e. $\psi \circ \iota = \iota'$.

Definition

Given two GHDs (M, g, ι) and (M', g', ι') of initial data $(\overline{M}, \bar{g}, \bar{k})$, we say that a GHD $(U \subseteq M, g|_U, \iota)$ is a *common globally hyperbolic development* (CGHD) of (M, g, ι) and (M', g', ι') iff (M', g', ι') is an extension of (U, g_U, ι_U).

The initial value problem in general relativity: definitions II

Definition

Given two GHDs (M, g, ι) and (M', g', ι') of the same initial data, we say that (M', g', ι') is an *extension* of (M, g, ι) iff there exists a time orientation preserving isometric embedding $\psi : M \to M'$ that preserves the initial data, i.e. $\psi \circ \iota = \iota'$.

Definition

Given two GHDs (M, g, ι) and (M', g', ι') of initial data $(\overline{M}, \bar{g}, \bar{k})$, we say that a GHD $(U \subseteq M, g|_U, \iota)$ is a *common globally hyperbolic development* (CGHD) of (M, g, ι) and (M', g', ι') iff (M', g', ι') is an extension of (U, g_U, ι_U).

The initial value problem in general relativity: local theory

Theorem (Choquet-Bruhat '52)

Given initial data for the vacuum Einstein equations, there exists a GHD, and for any two GHDs of the same initial data, there exists a CGHD.

Obstructions to transferring previous proof of the existence of a unique MGHD to the Einstein equations

The Einstein equations: qualitative differences

We do not have a fixed background.

- $\implies$ What do we mean by global uniqueness?
- $\implies$ How to take the union of all GHDs in the construction of the MGHD?

The Einstein equations: qualitative differences

We do not have a fixed background.

- $\Longrightarrow$ What do we mean by global uniqueness?
- $\Longrightarrow$ How to take the union of all GHDs in the construction of the MGHD?

The Einstein equations: qualitative differences

We do not have a fixed background.

- $\Longrightarrow$ What do we mean by global uniqueness?
- $\Longrightarrow$ How to take the union of all GHDs in the construction of the MGHD?

The idea of the old proof by Choquet-Bruhat and Geroch

Step 1: Showing the existence of a maximal element

Consider the 'set' P of all GHDs of given initial data.

Define a partial order $\leq$ on P by

$$M' \leq M \quad :\Longleftrightarrow \quad M \text{ is an extension of } M'.$$

In this way we obtain a partially ordered set $(P, \leq)$.

Show that every chain has an upper bound in P. Zorn's lemma then ensures that there exists at least one maximal element $M_{\max}$ in P.

Step 1: Showing the existence of a maximal element

Consider the 'set' P of all GHDs of given initial data.

Define a partial order $\leq$ on P by

$$M' \leq M \;:\Longleftrightarrow\; M \text{ is an extension of } M'.$$

In this way we obtain a partially ordered set $(P, \leq)$.

Show that every chain has an upper bound in P. Zorn's lemma then ensures that there exists at least one maximal element $M_{\max}$ in P.

Step 1: Showing the existence of a maximal element

Consider the 'set' P of all GHDs of given initial data.

Define a partial order $\leq$ on P by

$$M' \leq M \;:\Longleftrightarrow\; M \text{ is an extension of } M'.$$

In this way we obtain a partially ordered set $(P, \leq)$.

Show that every chain has an upper bound in P. Zorn's lemma then ensures that there exists at least one maximal element $M_{\max}$ in P.

Step 1: Showing the existence of a maximal element

Consider the 'set' P of all GHDs of given initial data.

Define a partial order $\leq$ on P by

$$M' \leq M \;:\Longleftrightarrow\; M \text{ is an extension of } M'.$$

In this way we obtain a partially ordered set $(P, \leq)$.

Show that every chain has an upper bound in P. Zorn's lemma then ensures that there exists at least one maximal element M_{max} in P.

Step 2: Showing that $M_{\max}$ is the MGHD

We need to show that any GHD M satisfies $M \leq M_{\max}$.

Glue M and $M_{\max}$ together along the maximal common globally hyperbolic development U, i.e., consider

$$\tilde{M} := (M \sqcup M_{\max})/_{\sim}.$$

Here, the equivalence relation $\sim$ is generated by $M \ni p \sim q \in M_{\max}$ if $p \in U \subseteq M$ and $\psi(p) = q$. Here, $\psi : U \hookrightarrow M_{\max}$ is the isometric embedding.

Claim: $\tilde{M}$ is Hausdorff.

Step 2: Showing that $M_{\max}$ is the MGHD

We need to show that any GHD M satisfies $M \leq M_{\max}$.

Glue M and $M_{\max}$ together along the maximal common globally hyperbolic development U, i.e., consider

$$\tilde{M} := \left(M \sqcup M_{\max}\right)/_{\sim}.$$

Here, the equivalence relation $\sim$ is generated by $M \ni p \sim q \in M_{\max}$ if $p \in U \subseteq M$ and $\psi(p) = q$. Here, $\psi : U \hookrightarrow M_{\max}$ is the isometric embedding.

Claim: $\tilde{M}$ is Hausdorff.

Step 2: Showing that $M_{\max}$ is the MGHD

We need to show that any GHD M satisfies $M \leq M_{\max}$.

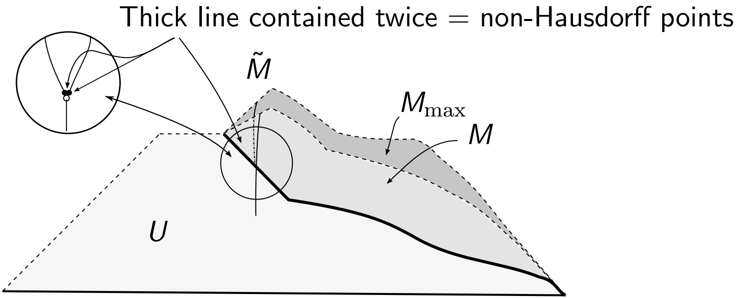
Glue M and $M_{\max}$ together along the maximal common globally hyperbolic development U, i.e., consider

$$\tilde{M} := \big(M \sqcup M_{\max}\big)/_{\sim}.$$

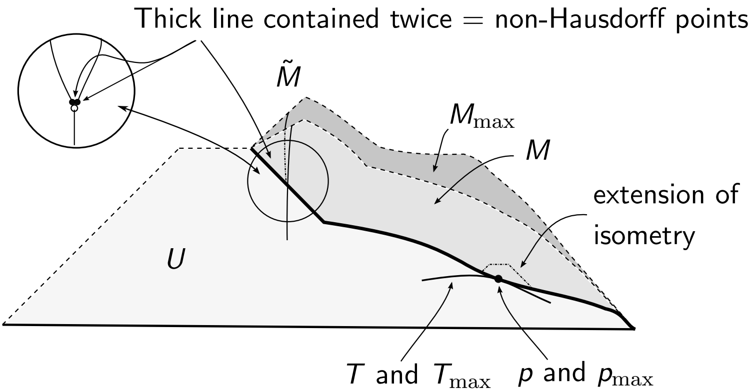
Here, the equivalence relation $\sim$ is generated by $M \ni p \sim q \in M_{\max}$ if $p \in U \subseteq M$ and $\psi(p) = q$. Here, $\psi : U \hookrightarrow M_{\max}$ is the isometric embedding.

Claim: $\tilde{M}$ is Hausdorff.

Step 2: Showing that $M_{\max}$ is the MGHD



Step 2: Showing that M_{max} is the MGHD



The new proof

Reinterpreting 'global uniqueness'

Global uniqueness $:\Longleftrightarrow$ Given two GHDs of the same initial data, then they agree on the intersection of their domains.

$:\Longleftrightarrow$ Given two GHDs of the same initial data, there exists a GHD that is an extension of both.

Reinterpreting 'global uniqueness'

Global uniqueness $:\iff$ Given two GHDs of the same initial data, then they agree on the intersection of their domains.

$:\iff$ Given two GHDs of the same initial data, there exists a GHD that is an extension of both.

Step 1 of the new proof: constructing the extension

Go back to the case of a quasilinear wave equation. There we would construct the extension $\tilde{M}$ by taking the union of M and M'.

The same result is obtained if we *glue* M and M' together along $M \cap M'$.

This is the same as glueing M and M' together along their MCGHD.

This construction carries over to the Einstein equations!! Note that it is the same procedure as encountered in the old proof - but there it was *not* used in order to construct bigger GHDs!

Summary

In the case of the Einstein equations, the appropriate analogue of 'taking the union' of two GHDs is to glue them together along their MCGHD.

Step 1 of the new proof: constructing the extension

Go back to the case of a quasilinear wave equation. There we would construct the extension $\tilde{M}$ by taking the union of M and M'.

The same result is obtained if we *glue* M and M' together along $M \cap M'$.

This is the same as glueing M and M' together along their MCGHD.

This construction carries over to the Einstein equations!! Note that it is the same procedure as encountered in the old proof - but there it was *not* used in order to construct bigger GHDs!

Summary

In the case of the Einstein equations, the appropriate analogue of 'taking the union' of two GHDs is to glue them together along their MCGHD.

Step 1 of the new proof: constructing the extension

Go back to the case of a quasilinear wave equation. There we would construct the extension $\tilde{M}$ by taking the union of M and M'.

The same result is obtained if we *glue* M and M' together along $M \cap M'$.

This is the same as glueing M and M' together along their MCGHD.

This construction carries over to the Einstein equations!! Note that it is the same procedure as encountered in the old proof - but there it was *not* used in order to construct bigger GHDs!

Summary

In the case of the Einstein equations, the appropriate analogue of 'taking the union' of two GHDs is to glue them together along their MCGHD.

Step 1 of the new proof: constructing the extension

Go back to the case of a quasilinear wave equation. There we would construct the extension $\tilde{M}$ by taking the union of M and M'.

The same result is obtained if we *glue* M and M' together along $M \cap M'$.

This is the same as glueing M and M' together along their MCGHD.

This construction carries over to the Einstein equations!! Note that it is the same procedure as encountered in the old proof - but there it was *not* used in order to construct bigger GHDs!

Summary

In the case of the Einstein equations, the appropriate analogue of 'taking the union' of two GHDs is to glue them together along their MCGHD.

Step 1 of the new proof: constructing the extension

Go back to the case of a quasilinear wave equation. There we would construct the extension $\tilde{M}$ by taking the union of M and M'.

The same result is obtained if we *glue* M and M' together along $M \cap M'$.

This is the same as glueing M and M' together along their MCGHD.

This construction carries over to the Einstein equations!! Note that it is the same procedure as encountered in the old proof - but there it was *not* used in order to construct bigger GHDs!

Summary

In the case of the Einstein equations, the appropriate analogue of 'taking the union' of two GHDs is to glue them together along their MCGHD.

Step 2 of the new proof: the construction of the MGHD

Idea: 'Take union of all GHDs', i.e., glue them together along their MCGHDs.

Consider the set $\{M_\alpha \mid \alpha \in A\}$ of all GHDs whose underlying differential manifold is a submanifold of $\mathbb{R} \times \overline{M}$.

Denote the MCGHD of M_{α_i} and M_{α_k} with $U_{\alpha_i\alpha_k} \subseteq M_{\alpha_i}$ and the corresponding isometric embedding with $\psi_{\alpha_i\alpha_k} : U_{\alpha_i\alpha_k} \to M_{\alpha_k}$.

Define an equivalence relation $\sim$ on $\bigsqcup_{\alpha\in A} M_\alpha$ by

$$M_{\alpha_i} \ni p_{\alpha_i} \sim q_{\alpha_k} \in M_{\alpha_k} \text{ iff } p_{\alpha_i} \in U_{\alpha_i\alpha_k} \text{ and } \psi_{\alpha_i\alpha_k}(p_{\alpha_i}) = q_{\alpha_k}$$

and take the quotient $(\bigsqcup_{\alpha\in A} M_\alpha)/_\sim =: \tilde{M}$ with the quotient topology.

Step 2 of the new proof: the construction of the MGHD

Idea: 'Take union of all GHDs', i.e., glue them together along their MCGHDs.

Consider the set $\{M_\alpha \,|\, \alpha \in A\}$ of all GHDs whose underlying differential manifold is a submanifold of $\mathbb{R} \times \overline{M}$.

Denote the MCGHD of M_{α_i} and M_{α_k} with $U_{\alpha_i\alpha_k} \subseteq M_{\alpha_i}$ and the corresponding isometric embedding with $\psi_{\alpha_i\alpha_k} : U_{\alpha_i\alpha_k} \to M_{\alpha_k}$.

Define an equivalence relation $\sim$ on $\bigsqcup_{\alpha\in A} M_\alpha$ by

$$M_{\alpha_i} \ni p_{\alpha_i} \sim q_{\alpha_k} \in M_{\alpha_k} \text{ iff } p_{\alpha_i} \in U_{\alpha_i\alpha_k} \text{ and } \psi_{\alpha_i\alpha_k}(p_{\alpha_i}) = q_{\alpha_k}$$

and take the quotient $(\bigsqcup_{\alpha\in A} M_\alpha)/_\sim =: \tilde{M}$ with the quotient topology.

Why a new proof?

- Zorn's lemma is not constructive. It guarantees the existence of the MGHD, but it doesn't give a procedure to obtain/construct it.
- Some people feel uneasy about the axiom of choice (which is equivalent to Zorn's lemma), since it entails unintuitive consequences like the Banach-Tarski paradox.
- Which mathematical axioms are actually needed for general relativity?
- The new proof fits in the previous natural framework for proving global uniqueness and the existence of a MGHD for a wide range of evolution equations.

- Zorn's lemma is not constructive. It guarantees the existence of the MGHD, but it doesn't give a procedure to obtain/construct it.
- Some people feel uneasy about the axiom of choice (which is equivalent to Zorn's lemma), since it entails unintuitive consequences like the Banach-Tarski paradox.
- Which mathematical axioms are actually needed for general relativity?
- The new proof fits in the previous natural framework for proving global uniqueness and the existence of a MGHD for a wide range of evolution equations.

- Zorn's lemma is not constructive. It guarantees the existence of the MGHD, but it doesn't give a procedure to obtain/construct it.
- Some people feel uneasy about the axiom of choice (which is equivalent to Zorn's lemma), since it entails unintuitive consequences like the Banach-Tarski paradox.
- Which mathematical axioms are actually needed for general relativity?
- The new proof fits in the previous natural framework for proving global uniqueness and the existence of a MGHD for a wide range of evolution equations.

- Zorn's lemma is not constructive. It guarantees the existence of the MGHD, but it doesn't give a procedure to obtain/construct it.
- Some people feel uneasy about the axiom of choice (which is equivalent to Zorn's lemma), since it entails unintuitive consequences like the Banach-Tarski paradox.
- Which mathematical axioms are actually needed for general relativity?
- The new proof fits in the previous natural framework for proving global uniqueness and the existence of a MGHD for a wide range of evolution equations.