The L^p Dirichlet problem for perturbation operators on rough domains

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Classical Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in C(\partial\Omega)$ does there exists u satisfying

$$\begin{aligned} Lu = \operatorname{div}(A(X)\nabla u) &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \text{ ?} \end{aligned} \tag{1}$$

$A(X) = (a_{ij}(X))$ is symmetric measurable and satisfies

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(X)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } X \in \Omega \textit{ and } \xi \in \mathbb{R}^n.$$

- Ω is regular for L, if $\forall f \in C(\partial\Omega)$, $u_f = u \in C(\overline{\Omega})$.
- If Ω is regular there is a family probability measures $\{\omega_L^X\}_{X\in\Omega}$

$$u(X) = \int_{\partial\Omega} f(Q)d\omega_L^X(Q).$$

- ω_L^X is called the $L-$elliptic measure of Ω with pole X.

If Ω is regular how *smooth* is u?

- The interior regularity is a classical result: u is Hölder continuous in Ω (De Giorgi-Nash-Moser).
- Additional regularity of A implies higher interior regularity of the solution.
- This is a question about boundary regularity.
- The boundary regularity of u is determined by the regularity of ω_L

Results on NTA domains - Jerison-Kenig

- NTA (non-tangentially accessible) domains are regular
- ω_L is doubling, i.e. there exists a constant $C > 0$ such that for all $Q \in \partial\Omega$ and $0 < r < \operatorname{diam}\Omega$
$$\omega_L(B(Q, 2r)) \leq C\omega_L(B(Q, r)).$$
- The non-tangential limit of the solution of (1) at the boundary exists and coincides with f ω_L-a.e (i.e $u = f$ on $\partial\Omega$ ω_L-a.e.).
- If f is Lipschitz, u is Hölder continuous in $\overline{\Omega}$.

Domains & Operators

- Domains
  1. Ω Lipschtiz
  2. Ω chord arc domain (CAD)

- Operators
  1. $L_0 = \Delta$
  2. L_1 perturbation of "good" operator L_0

Domains

- $\Omega \subset \mathbb{R}^n$ is a chord arc domain (CAD) is Ω is a NTA whose boundary is Ahlfors regular, i.e. for $Q \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$

$$\sigma(\Delta(Q,r) \sim r^{n-1}.$$

Here σ is the surface measure of $\partial\Omega$ and $\Delta(Q,r) = B(Q,r) \cap \partial\Omega$.

- Examples:

$$\Omega = \{(\bar{x}, x_n) : x_n > \varphi(\bar{x})\}$$

  - φ Lipschitz
  - $\nabla\varphi \in BMO$

- Remarks:
  - Locally the boundary of a CAD is not necessarily the graph of a function.
  - CADs may appear as Hausdorff limits of smooth domains.

$(D)_p$ problem on CAD

- Given $f \in L^p(\sigma)$ with $p > 1$ does there exist u such that

$$\begin{aligned} Lu &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{aligned}$$

with

$$\|N(u)\|_{L^p(\sigma)} \leq C\|f\|_{L^p(\sigma)} \quad \text{where} \quad N(u) = \sup_{X \in \Gamma(Q)} |u(X)|\,?$$

$$\Gamma(Q) = \{X \in \Omega : |X - Q| \leq 2\delta(X)\} \quad \text{and} \quad \delta(X) = \mathrm{dist}\,(\mathrm{X}, \partial\Omega)$$

$(D)_p$ problem & Weights

- If the $(D)_p$ problem holds then

$$\lim_{X \to Q, X \in \Gamma(Q)} u(X) = f(Q) \quad \text{for} \quad \sigma - a.e.\ Q \in \partial\Omega.$$

- The $(D)_p$ holds for L if and only if $\omega_L \in B_q(\sigma)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

- $\omega_L \in B_q(\sigma)$ if $k_L = \frac{d\omega_L}{d\sigma}$ satisfies

$$\left(\frac{1}{\sigma(\Delta)} \int_\Delta k_L^q \, d\sigma \right)^{\frac{1}{q}} \leq C \frac{1}{\sigma(\Delta)} \int_\Delta k_L \, d\sigma$$

for all $\Delta = B(Q, r) \cap \partial\Omega$ with $Q \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$.

- Note $B_q(\sigma) \subset B_{q'}(\sigma)$ for $q' < q$. Recall $A_\infty(\sigma) = \cup_{p>1} B_p(\sigma)$.

$(D)_p$ problem for the Laplacian

- Lipschitz domains -Dahlberg.
  - The harmonic measure $\omega \in A_\infty(\sigma)$ and $\omega \in B_2(\sigma)$
  - On Lipschitz domains the $(D)_p$ problem for the Laplacian can be solved for $p \geq 2$.

- David-Jerison & Semmes: If Ω is a CAD there exists $q \in (1, \infty)$ such that $\omega \in B_q(\sigma)$. Given $q > 1$ there exists a CAD, Ω, such that $\omega \notin B_q(\sigma)$.

What do we know about the regularity of the elliptic measure for general divergence form elliptic operators?

- Caffarelli-Fabes-Kenig & Modica-Mortola constructed operators L on smooth domains for which ω_L and σ are mutually singular.

Question: *Characterize the operators L for which $\omega_L \in B_q(\sigma)$ for some $q \in (1, \infty)$.*

The behavior of A near $\partial\Omega$ determines the regularity of ω_L with respect to σ.

Remark: Let $L_0 = \mathrm{div}\,(A_0(X)\,\nabla)$ and $L_1 = \mathrm{div}\,(A_1(X)\,\nabla)$ be symmetric elliptic operators. Assume A_0 and A_1 coincide in a neighborhood of $\partial\Omega$ then $k_0 \in B_q(\sigma)$ if and only if $k_1 \in B_q(\sigma)$.

- L_1 is a **perturbation** of L_0 if the deviation function

$$a(X) = \sup\{|A_1(Y) - A_0(Y)| : Y \in B(X, \delta(X)/2)\}$$

satisfies $\frac{a^2(X)}{\delta(X)}\,dX$ is a Carleson measure.

$$\frac{a^2(X)}{\delta(X)}\, dX \text{ is a Carleson measure in } \Omega$$

if there exists $C > 0$ such that for all $Q \in \partial\Omega$ and $r > 0$

$$h(Q,r) = \left(\frac{1}{\sigma(B(Q,r))} \int_{T(Q,r)} \frac{a^2(X)}{\delta(X)}\, dX \right)^{1/2} < C.$$

$T(Q,r) = B(Q,r) \cap \Omega$ is the Carleson region associated to the surface ball $B(Q,r) \cap \partial\Omega$.

Remark: If L_1 is a perturbation of L_0 then $A_1 = A_0$ on $\partial\Omega$.

Perturbation operators on Lipschitz domains

- Dahlberg: Assume

$$\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r)=0.$$

Then $\omega_0 \in B_p(\sigma)$, for some $p > 1$, if and only if $\omega_1 \in B_p(\sigma)$.

General results for perturbation operators

- Fefferman-Kenig-Pipher: Let Ω be a Lipschitz domain. Assume

$$\sup_{Q\in\partial\Omega}\sup_{r>0} h(Q,r) < C$$

Then $\omega_0 \in A_\infty(\sigma)$ if and only if $\omega_1 \in A_\infty(\sigma)$.

- Milakis-Pipher-Toro: Let Ω be a chord arc domain. Assume

$$\sup_{Q\in\partial\Omega}\sup_{r>0} h(Q,r) < C$$

Then $\omega_0 \in A_\infty(\sigma)$ if and only if $\omega_1 \in A_\infty(\sigma)$.

Main Tool

- Fefferman-Kenig-Pipher. Let Ω be a Lipschitz domain, there is $\epsilon_0 > 0$ so that if

$$\sup_{\Delta}\left(\frac{1}{\omega_0(\Delta)}\int_{T(\Delta)} a^2(X)\frac{G_0(X)}{\delta^2(X)}dX\right)^{1/2} < \epsilon_0$$

then $\omega_1 \in B_2(\omega_0)$. Here $G_0(X)$ denotes the Green's function for L_0, $\Delta = B(Q,r)\cap\partial\Omega$ with $Q \in \partial\Omega$ and $r > 0$, and $T(\Delta) = B(Q,r)\cap\Omega$.

- Milakis-Pipher-Toro. Let Ω be a CAD, there is $\epsilon_0 > 0$ so that if

$$\sup_{\Delta}\left(\frac{1}{\omega_0(\Delta)}\int_{T(\Delta)} a^2(X)\frac{G_0(X)}{\delta^2(X)}dX\right)^{1/2} < \epsilon_0$$

then $\omega_1 \in B_2(\omega_0)$.

Why is this the main tool?

- Let Ω be a CAD. Assume $\omega_0 \in A_\infty(\sigma)$, then given $\epsilon > 0$ there exists $\delta > 0$ such that if

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2} \leq \delta,$$

then

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon.$$

Thus $\omega_1 \in B_2(\omega_0)$ which combined with $\omega_0 \in A_\infty(\sigma)$ yields $\omega_1 \in A_\infty(\sigma)$.

Lipschitz domains vs chord arc domains

Differences & difficulties:

- Lack of a local representation for the boundary of CAD as a graph.
- Lack of results concerning approximation of CAD by interior smooth CAD.

Tools:

- Estimate for the Green function on NTA domains.
- The geometry of the NTA domain and the Ahlfors regularity of the surface measure to the boundary ensure that integration over Carleson regions can be handled by Fubini as it was in half space (GMT).
- Harmonic analysis in CAD: We developed the theory of tent spaces for functions defined on CAD. Tent spaces for functions defined on half space were initially studied by Coifman-Meyer-Stein.

Why did we start thinking about this problem?

- Escauriaza: Let Ω be a Lipschitz domain. Assume that

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0.$$

where

$$h(Q,r) = \left(\frac{1}{\sigma(\Delta(Q,r))} \int_{T(\Delta(Q,r))} \frac{a^2(X)}{\delta(X)} dX \right)^{1/2}.$$

Then $\log k_0 \in VMO(\sigma)$ if and only if $\log k_1 \in VMO(\sigma)$ where $k_j = \frac{d\omega_j}{d\sigma}$.

- Escauriaza & Jerison-Kenig: Let Ω be a C^1 domain, $L_0 = \Delta$ and assume that

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0.$$

then $\log k_1 \in VMO(\sigma)$.

Motivating question

- Kenig & Toro: If Ω is a CAD whose boundary is Reifenberg flat and such that the unit normal to the boundary $\overrightarrow{n} \in VMO(\sigma)$ (CAD with vanishing constant) then the harmonic measure and $L_0 = \Delta$

$$\log k_0 \in VMO(\sigma)$$

- Let Ω be a CAD with vanishing constant. Assume that $L_0 = \Delta$ and that

$$\lim_{r \to 0} \sup_{Q \in \partial\Omega} h(Q, r) = 0.$$

does $\log k_1 \in VMO(\sigma)$?

Once again CADs behave like Lipschitz domains

- Milakis-Pipher-Toro: Let Ω be a CAD. Assume that

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0.$$

Then $\log k_0 \in VMO(\sigma)$ if and only if $\log k_1 \in VMO(\sigma)$.

- How does this relate to the previous results?
  - $\log k \in VMO(\sigma)$ if and only if $\omega \in B_q(\sigma)$ for $q > 1$ and

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} \left(\fint_{B(Q,r)} k^q \, d\sigma \right)^{\frac{1}{q}} \left(\fint_{B(Q,r)} k \, d\sigma \right)^{-1} = 1$$

Lipschitz vs chord arc domains

- On a Lipschitz domain if $\log k_0 \in VMO(\sigma)$ and $\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r) = 0$ then Dahlberg's result ensure that $\omega_1 \in B_2(\sigma)$. Escauriaza showed that an optimal B_2 inequality holds.

- What did we know?

- On a CAD if $\log k_0 \in VMO(\sigma)$ and $\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r) = 0$ then $\omega_1 \in A_\infty(\sigma)$, i.e. $\exists q > 1$ such that $\omega_1 \in B_q(\sigma)$ (Milakis-Toro).

- Was this enough? NO

- Results in [MPT] ensure that on a CAD if $\log k_0 \in VMO(\sigma)$ and $\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r) = 0$ then $\omega_1 \in B_2(\sigma)$.

Regularity results for small perturbation operators - MPT

- Let Ω be a CAD if $\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0$ and $\log k_0 \in VMO(\sigma)$ then

$$\left(\fint_{B(Q,r)} k_1^2 \, d\sigma \right)^{\frac{1}{2}} \left(\fint_{B(Q,r)} k_1 \, d\sigma \right)^{-1} \leq Cr^\gamma + Ch(Q,r)$$

$$+ \left(\fint_{B(Q,r)} k_0^2 \, d\sigma \right)^{\frac{1}{2}} \left(\fint_{B(Q,r)} k_0 \, d\sigma \right)^{-1}$$

Thus $\log k_1 \in VMO(\sigma)$.

Sketch of the proof: Dahlberg's idea

For $t \in [0,1]$ consider the operators

$$\begin{aligned} L_t u &= \mathrm{div}(A_t \nabla u) \\ A_t(X) &= (1-t)A_0(X) + tA_1(X). \end{aligned}$$

Let ω_t be the elliptic measure of L_t and $k_t = \frac{d\omega_t}{d\sigma}$.For $Q \in \partial\Omega$ and$r > 0$ let $\Delta_r = B(Q,r) \cup \partial\Omega$. For $f \in L^2(\sigma)$ let

$$\Psi(t) = \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} f\, k_t \, d\sigma.$$

Then $\Psi(t)$ is Lipschitz and

$$\dot{\Psi}(t) = \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} \dot{k}_t \left(f - \fint_{\Delta_r} f d\omega_t \right) d\sigma$$

where $\dot{k}_t$ is the weak L^2 limit of $(k_{t+h} - k_t)/h$ as h tends to zero.

Idea behind the proof

For $t \in [0,1]$ consider

$$\begin{aligned} L_t u_t &= \operatorname{div}(A_t \nabla u_t) \text{ in } \Omega \\ u_t &= f \text{ in } \partial\Omega \end{aligned}$$

For $t, s \in [0,1]$ and $\varepsilon(Y) = A_1(Y) - A_0(Y)$

$$u_s(X) - u_t(X) = (s-t) \int_\Omega \varepsilon(Y) \nabla G_t(X,Y) \nabla u_s(Y) dY.$$

$$\int_\Omega |\varepsilon(Y)| |\nabla G_t(X,Y)| |\nabla u_s(Y)| dY \lesssim ||f||_{L^2(\sigma)}$$

and

$$|u_s(X) - u_t(X)| \lesssim ||f||_{L^2(\sigma)} |s-t|.$$

Technical lemma

- There exist $\gamma, \beta \in (0,1)$ such that if $f \in L^2(\sigma)$, $f \geq 0$ and $\|f\|_{L^2(d\sigma/\sigma(\Delta_r))} \leq 1$ for $t \in [0,1]$

$$|\dot{\Psi}(t)| \leq C\left[r^\gamma + \sup_{s \leq r^\beta} \sup_{Q \in \partial\Omega} h(Q,s)\right]$$

- Integration guarantees that

$$\Psi(1) \leq \Psi(0) + C\left[r^\gamma + \sup_{s \leq r^\beta} \sup_{Q \in \partial\Omega} h(Q,s)\right]$$

- By duality

$$\frac{\sigma(\Delta_r)}{\omega_1(\Delta_r)}\left(⨍_{\Delta_r} k_1^2\, d\sigma\right)^{\frac{1}{2}} \leq \frac{\sigma(\Delta_r)}{\omega_0(\Delta_r)}\left(⨍_{\Delta_r} k_0^2\, d\sigma\right)^{\frac{1}{2}} + C\left[r^\gamma + \sup_{s \leq r^\beta} \sup_{Q \in \partial\Omega} h(Q,s)\right]$$