1. (a) Find the remainder when \( n^2 + 4 \) is divided by 7 for \( 0 \leq n < 7 \).

Deduce that \( n^2 + 4 \) is not divisible by 7, for every positive integer \( n \). [Hint: write \( n = 7k + r \) where \( 0 \leq r < 7 \).]

(b) Now \( k \) is an integer such that \( n^3 + k \) is not divisible by 4 for all integers \( n \). What are the possible values of \( k \)?

2. (i) Prove that if \( a, b \) are positive real numbers then

\[ \sqrt{ab} \leq \frac{1}{2}(a + b). \]

(ii) Now let \( a_1, a_2, \ldots, a_n \) be positive real numbers. Let \( S = a_1 + a_2 + \cdots + a_n \) and \( P = a_1a_2 \cdots a_n \).

Suppose that \( a_i \) and \( a_j \) are distinct. Show that replacing \( a_i \) and \( a_j \) with \((a_i + a_j)/2\) and \((a_i + a_j)/2\) increases \( P \) without changing \( S \).

Deduce that

\[ (a_1a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}. \]

3. (i) Let \( n \) be a positive integer. Show that

\[ x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \]

(ii) Let \( a \) also be a positive integer. Show that if \( a^n - 1 \) is prime then \( a = 2 \) and \( n \) is prime.

Is it true that if \( n \) is prime then \( 2^n - 1 \) is also prime?

4. Let \( a, b, r, s \) be rational numbers with \( s \neq 0 \). Suppose that the number \( r + s\sqrt{2} \) is a root of the quadratic equation

\[ x^2 + ax + b = 0. \]

Show that \( r - s\sqrt{2} \) is also a root.

5. (i) The cubic equation \( ax^3 + bx^2 + cx + d = 0 \) has roots \( \alpha, \beta, \gamma \), and so factorises as

\[ a(x - \alpha)(x - \beta)(x - \gamma). \]

Determine

\[ \alpha + \beta + \gamma, \quad \alpha\beta + \beta\gamma + \gamma\alpha, \quad \alpha\beta\gamma, \]

in terms of \( a, b, c, d \). What does \( \alpha^2 + \beta^2 + \gamma^2 \) equal?

(ii) Show that \( \cos 3\theta = 4\cos^3 \theta - 3\cos \theta \).

(iii) By considering the roots of the equation \( 4x^3 - 3x - \cos 3\theta = 0 \) deduce that

\[ \cos \theta \cos(\theta + 2\pi/3) \cos(\theta + 4\pi/3) = \frac{\cos(3\theta)}{4}. \]

What do \( \cos \theta + \cos(\theta + 2\pi/3) + \cos(\theta + 4\pi/3) \) and \( \cos^2 \theta + \cos^2(\theta + 2\pi/3) + \cos^2(\theta + 4\pi/3) \) equal?