ALGEBRA EXERCISES 2

1. Under what conditions on the real numbers a, b, c, d, e, f do the simultaneous equations

$$ax + by = e \quad \text{and} \quad cx + dy = f$$

have (a) a unique solution, (b) no solution, (c) infinitely many solutions in x and y.

Select values of a, b, c, d, e, f for each of these cases, and sketch on separate axes the lines $ax+by=e$ and $cx+dy=f$.

2. For what values of a do the simultaneous equations

$$\begin{aligned} x + 2y + a^2 z &= 0, \\ x + ay + z &= 0, \\ x + ay + a^2 z &= 0, \end{aligned}$$

have a solution other than $x = y = z = 0$. For each such a find the general solution of the above equations.

3. Do 2×2 matrices exist satisfying the following properties? Either find such matrices or show that no such exist.
(i) A such that $A^5 = I$ and $A^i \neq I$ for $1 \leq i \leq 4$,
(ii) A such that $A^n \neq I$ for all positive integers n,
(iii) A and B such that $AB \neq BA$,
(iv) A and B such that AB is invertible and BA is singular (i.e. not invertible),
(v) A such that $A^5 = I$ and $A^{11} = 0$.

4. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and let} \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

be a 2×2 matrix and its *transpose*. Suppose that $\det A = 1$ and

$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $a^2 + c^2 = 1$, and hence that a and c can be written as

$$a = \cos\theta \quad \text{and} \quad c = \sin\theta.$$

for some θ in the range $0 \leq \theta < 2\pi$. Deduce that A has the form

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

5. (a) Prove that

$$\det(AB) = \det(A)\det(B)$$

for any 2×2 matrices A and B.

(b) Let A denote the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that

$$A^2 - (\text{trace}A)A + (\det A)I = 0 \tag{1}$$

where

- $\text{trace}A = a + d$ is the trace of A, that is the sum of the diagonal elements;
- $\det A = ad - bc$ is the determinant of A;
- I is the 2×2 identity matrix.

(c) Suppose now that $A^n = 0$ for some $n \geq 2$. Prove that $\det A = 0$. Deduce using equation (1) that $A^2 = 0$.