Week 3 — Differential Equations

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Abstract

Linear differential equations with constant coefficients. Homogeneous and inhomogeneous equations. Integrating factors. Homogeneous polar equations.

1 Introduction

The study of ordinary differential equations (DEs)[1] is as old as The Calculus itself and dates back to the time of Newton (1643-1727) and Leibniz (1646-1716). At that time most of the interest in DEs came from applications in physics and astronomy — one of Newton's greatest achievements in his *Principia Mathematica* (1687) was to show that a force between a planet and the sun, which is inversely proportional to the square of the distance between them, would lead to an elliptical orbit. The study of differential equations grew as increasingly varied mathematical and physical situations led to differential equations, and as more and more sophisticated techniques were found to solve them — besides astronomy, DEs began appearing naturally in applied areas such as fluid dynamics, heat flow, waves on strings, and equally in pure mathematics, in determining the curve a chain between two points will make under its own weight, the shortest path between two points on a surface, the surface across a given boundary of smallest area (i.e. the shape of a soap film), the largest area a rope of fixed length can bound, etc.

We give here, and solve, a simple example which involves some of the key ideas of DEs — the example here is the movement of a particle P under gravity, in one vertical dimension. Suppose that we write $h(t)$ for the height (in metres, say) of P over the ground; if the heights involved are small enough then we can reasonably assume that the gravity acting (denoted as g) is constant, and so we have

$$\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -g. \tag{1}$$

The *velocity* of the particle is the quantity $\mathrm{d}h/\mathrm{d}t$ — the rate of change of distance (here, height) with time. The rate of change of velocity with time is called *acceleration* and is the quantity $\mathrm{d}^2h/\mathrm{d}t^2$ on the LHS of the above equation. The acceleration here is entirely due to gravity. Note the need for a minus sign here as gravity is acting downwards.

Equation (1) is not a difficult DE to solve; we can integrate first once,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -gt + K_1 \tag{2}$$

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[1] We will be studying here *ordinary differential equations* (ODEs) rather than *partial differential equations* (PDEs). This means that the DEs in question will involve *full* derivatives, such as $\mathrm{d}y/\mathrm{d}x$, rather than *partial* derivatives, such as $\partial y/\partial x$. The latter notation is a measure of how a function y changes with x whilst all other variables (which y depends on) are kept constant.

and then again

$$h(t) = \frac{-1}{2} g t^2 + K_1 t + K_2 \tag{3}$$

where K_1 and K_2 are constants. Currently we don't know enough about the specific case of the particle P to be able to say anything more about these constants. Note though that whatever the values of K_1 and K_2 the path of P is a parabola.

Remark 1 *Equation (1) is a second order DE. A derivative of the form $d^k y/dx^k$ is said to be of order k and we say that a DE has order k if it involves derivatives of order k and less. In some sense solving a DE of order k involves integrating k times, though not usually in such an obvious fashion as in the DE above. So we would expect the solution of an order k DE to have k undetermined constants in it, and this will be the case in most of the simple examples that we look at here. However this is not generally the case and we will see other examples where more, or fewer, than k constants are present in the solution.*

So the *general solution (3)* for $h(t)$ is not unique, but rather depends on two constants. And this isn't unreasonable as the particle P could follow many a path; at the moment we don't have enough information to characterise the path uniquely. One way of filling in the missing info would be to say how high P was at $t = 0$ and how fast it was going at that point. For example, suppose P started at a height of 100m and we threw it up into the air at a speed of 10ms^{-1} — that is

$$h(0) = 100 \quad \text{and} \quad \frac{\mathrm{d}h}{\mathrm{d}t}(0) = 10. \tag{4}$$

Then putting these values into equations (2) and (3) we'd get

$$10 = \frac{\mathrm{d}h}{\mathrm{d}t}(0) = -g \times 0 + K_1 \quad \text{giving } K_1 = 10,$$

and

$$100 = h(0) = \frac{-1}{2} g \times 0^2 + K_1 \times 0 + K_2 \quad \text{giving } K_2 = 100.$$

So the height of P at time t has been uniquely determined and is given by

$$h(t) = 100 + 10t - \frac{1}{2} g t^2.$$

The extra bits of information given in equation (4) are called *initial conditions* — particles like P can travel along infinitely many paths (all of them parabolae), but we need extra information to identify this path exactly.

Having solved the DE and found an equation for h then we could easily answer other questions about P's behaviour such as

- what is the greatest height P achieves? The maximum height will be a stationary value for $h(t)$ and so we need to solve the equation $h'(0) = 0$, which has solution $t = 10/g$. At this time the height is

$$h(10/g) = 100 + \frac{100}{g} - \frac{100g}{2g^2} = 100 + \frac{50}{g}.$$

- what time does P hit the ground? To solve this we see that

$$0 = h(t) = 100 + 10t - \frac{1}{2} g t^2$$

has solutions

$$t = \frac{-10 \pm \sqrt{100 + 200g}}{-g}.$$

One of these times is meaningless (being negative, and so before our experiment began) and so we take the other (positive) solution and see that P hits the ground at

$$t = \frac{10 + 10\sqrt{1 + 2g}}{g}.$$

We end this section with an example by way of warning of doing anything too cavalier with DEs.

Example 2 *Find the general solution of the DE*

$$\left(\frac{dy}{dx}\right)^2 = 4y. \tag{5}$$

Given this equation we might argue as follows — taking square roots we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{y}, \tag{6}$$

which we would recognise as a *separable* DE. This means we can separate the variables onto either side of the DE to get

$$\frac{\mathrm{d}y}{2\sqrt{y}} = \mathrm{d}x, \tag{7}$$

perhaps not worrying too much about what this formally means, and then we can integrate this to get

$$\sqrt{y} = \int \frac{\mathrm{d}y}{2\sqrt{y}} = \int \mathrm{d}x = x + K,$$

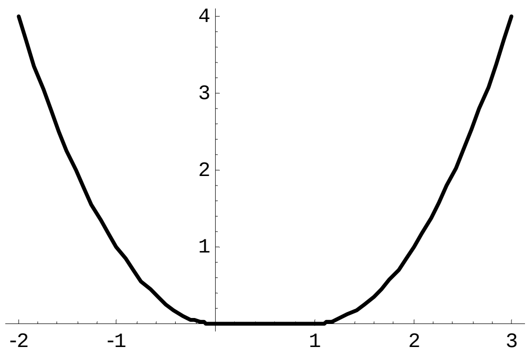
where K is a constant. Squaring this, we might think that the general solution has the form

$$y = (x+K)^2\,.$$

What, if anything, could have gone wrong with this argument? We could have been more careful to include positive and negative square roots at the (6) stage, but actually we don't lose any solutions by this oversight. Thinking a little more, we might realise that we have missed the most obvious of solutions: the zero function, $y = 0,$.which isn't present in our 'general' solution. At this point we might scold ourselves for committing the greatest crime known in mathematics at stage (7) by dividing by zero, rather than treating $y = 0$ as a separate case. But we have lost many more than just one solution at this point here by being careless. The general solution of (5) is in fact

$$y(x) = \begin{cases} (x-a)^2 & x \leq a \\ 0 & a \leq x \leq b \\ (x-b)^2 & b \leq x \end{cases}$$

where a and b are constants satisfying $-\infty \leq a \leq b \leq \infty$. We missed whole families of solutions by being careless — note also that the general solution requires *two* constants in its descrition even though the DE is only first order.



A missed solution with $a = 0$ and $b = 1$

2 Linear Differential Equations

A *homogeneous linear* differential equation is a DE with the following properties

- if y_1 and y_2 are solutions of the DE then so is $y_1 + y_2$;
- if y is a solution of the DE and c is a constant then cy is also a solution.

So, the following are examples of linear DEs

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + y &= 0, \\ x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - 1\right) y &= 0, \end{aligned}$$

whilst the following two are not

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 1, \tag{8}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0. \tag{9}$$

Note that it is easy to check the first two DEs are linear even though solving the second equation is no routine matter. To see that the second two DEs aren't linear we could note that:

- 1 and $e^{-x} + 1$ are solutions of equation (8) though their sum $e^{-x} + 2$ is not a solution.
- x^{-1} is a solution of equation (9) though $2x^{-1}$ is not a solution.

A homogeneous linear DE, of order k, involving a function y of a single variable x has, as its most general form

$$f_k(x) \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + f_{k-1}(x) \frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + f_1(x) \frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)\, y = 0. \tag{10}$$

The word 'homogeneous' here refers to the fact that there is a zero on the RHS of the above equation. An *inhomogeneous* linear DE of order k, involving a function y of a single variable x is one of the form

$$f_k(x) \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + f_{k-1}(x) \frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + f_1(x) \frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)\, y = g(x). \tag{11}$$

Remark 3 *Solving the two equations are very closely linked because of the linear algebra behind their solution. The solutions of equation (10) form a real vector space, (cf. first term linear algebra course) the dimension of which is the number of independent constants present in the general solution of the equation. The solutions of equation (11) form a real affine space of the same dimension, and its vector space of translations is just the solution space of (10).*

What all this technical verbiage means is that if $y(x)$ is a solution of the homogeneous equation (10) and $Y(x)$ is a solution of the inhomogeneous equation (11) then $Y(x) + y(x)$ is also a solution of (11) — and importantly every solution of (11) is ot the form $Y(x) + y(x)$ — what this boils down to, is that solving (11) amounts to knowing how to solve (10) and being able to find any particular solution $Y(x)$ of (11). (From a geometrical point of view the solution spaces of (10) and (11) are parallel; we just need some point $Y(x)$ in the second solution space and from there we move around just as before.) The meaning of this remark will become clearer as we go through some examples.

2.1 Homogeneous Equations with Constant Coefficients

We begin by considering homogeneous linear DEs where the functions $f_i(x)$ are all constant.

Theorem 4 *Consider the DE*

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

with auxiliary equation (AE)

$$m^2 + Am + B = 0.$$

The general solution to the DE is:

1. *in the case when the AE has two distinct real solutions α and β:*

$$Ae^{\alpha x} + Be^{\beta x};$$

2. *in the case when the AE has a repeated real solution α:*

$$(Ax + B)\, e^{\alpha x};$$

3. *in the case when the AE has a complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$:*

$$e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right).$$

Remark 5 *For those who have not met complex numbers before the following proof may be somewhat difficult. It is possible to get a good sense of the proof by assuming that λ and μ (as defined below) are both real, which is all that is needed to understand a proof of the first two cases. For the proof to make sense in the third case some understanding of complex numbers and the relation*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is necessary. For more on this see the handout accompanying the week 4 talk on complex numbers.

Proof. Let's call the roots of the AE λ and μ, at the moment allowing for any of the above cases to hold. We can rewrite the original DE as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (\lambda + \mu)\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\mu y = 0.$$

We will make the substitution

$$z(x) = y(x)\, e^{-\mu x},$$

noting that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{\mu x}\right) = \frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} + \mu z e^{\mu x},$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\mu x} + 2\mu\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} + \mu^2 z e^{\mu x}.$$

Hence our original DE, as an new DE involving $z(x)$, has become

$$\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\mu x} + 2\mu\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} + \mu^2 z e^{\mu x}\right) - (\lambda + \mu)\left(\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} + \mu z e^{\mu x}\right) + \lambda\mu z e^{\mu x} = 0,$$

which simplifies to

$$\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\mu x} + 2\mu\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x}\right) - (\lambda + \mu)\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} = 0.$$

Dividing through by $e^{\mu x}$ gives

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} + (\mu - \lambda)\frac{\mathrm{d}z}{\mathrm{d}x},$$

and we can simplify this further by substituting

$$w(x) = \frac{\mathrm{d}z}{\mathrm{d}x}$$

to get

$$\frac{\mathrm{d}w}{\mathrm{d}x} = (\lambda - \mu)\, w. \tag{12}$$

We now have two cases to consider: when $\lambda = \mu$ and when $\lambda \neq \mu$.

In the case when the roots are equal then (12) leads to the following line of argument

$$\begin{aligned} w(x) &= \frac{\mathrm{d}z}{\mathrm{d}x} = A \quad \text{(a constant)}, \\ z(x) &= Ax + B \quad (A \text{ and } B \text{ constants}), \\ y(x) &= z(x)\, e^{\mu x} = (Ax + B)\, e^{\mu x}, \end{aligned}$$

as we stated in case 2 of the theorem.

In the case when the roots are distinct (either real or complex) then (12) has solution

$$w(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = c_1 e^{(\lambda - \mu)x}$$

(where c_1 is a constant) and so integrating gives

$$z(x) = \frac{c_1}{\lambda - \mu} e^{(\lambda - \mu)x} + c_2$$

(where c_2 is a second constant) to finally find

$$y(x) = z(x)\, e^{\mu x} = \frac{c_1}{\lambda - \mu} e^{\lambda x} + c_2 e^{\mu x}.$$

When λ and μ are real then this solution is in the required form for case 1 of the theorem. When $\lambda = \alpha + i\beta$ and $\mu = \alpha - i\beta$ are complex conjugates then this solution is in the correct form for case 3 of the theorem once we remember that

$$e^{(\alpha \pm i\beta)x} = e^{\alpha x}(\cos \beta x \pm i \sin \beta x).$$

■

Example 6 *Solve the equation*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0,$$

with initial conditions

$$y(0) = 1, y'(0) = 0.$$

This has auxiliary equation

$$0 = m^2 - 3m + 2 = (m - 1)(m - 2)$$

which has roots $m = 1$ and $m = 2$. So the general solution of the equation is

$$y(x) = Ae^{x} + Be^{2x}.$$

Now the initial conditions imply

$$\begin{aligned} 1 &= y(0) = A + B, \\ 0 &= y'(0) = A + 2B. \end{aligned}$$

Hence

$$A = 2 \text{ and } B = -1.$$

So the *unique* solution of this DE with initial solutions is

$$y(x) = 2e^x - e^{2x}.$$

The theory behind the solving of homogeneous linear DEs with constant coefficients extends to all orders, and not to second order DEs, provided suitable adjustments are made.

Example 7 *Write down the general solution of the following DE*

$$\frac{d^6y}{dx^6} + 2\frac{d^5y}{dx^5} + \frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 4y = 0$$

This has auxiliary equation

$$m^6 + 2m^5 + m^4 - 4m^3 - 4m^2 + 4 = 0.$$

With can see (with a little effort) that this factorises as

$$(m-1)^2 \left(m^2 + 2m + 2\right)^2 = 0$$

which has roots $1, -1+i$ and $-1-i$, all of which are repeated roots. So the general solution of the DE is

$$y(x) = (Ax + B)e^x + (Cx + D)e^{-x}\cos x + (Ex + F)e^{-x}\sin x.$$

2.2 Inhomogeneous Equations

The examples we discussed in the previous subsection were homogeneous — that is they had the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + A(x)\frac{\mathrm{d}y}{\mathrm{d}x} + B(x)y = 0, \tag{13}$$

and we concentrated on examples where the functions $A(x)$ and $B(x)$ were constants. Here we shall look at *inhomgeneous* examples of second order linear DEs with constant coefficients: that is those of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + A(x)\frac{\mathrm{d}y}{\mathrm{d}x} + B(x)y = f(x). \tag{14}$$

As commented in Remark (3) we have done much of the work already — what remains is just to find *a particular solution* of (14).

Proposition 8 *If $Y(x)$ is a solution of the inhomogeneous equation (14) and $y(x)$ is a solution of the homogeneous equation (13) then $Y(x) + y(x)$ is also a solution of the inhomogeneous equation. Indeed every solution of the inhomogeneous equation is of the form*

$$Y(x) + y(x)$$

where $Y(x)$ is a solution of the homogeneous equation.

Proof. The proof is routine and left as an exercise. ■

So we see that once we have solved (13) then solving (14) reduces to finding a single solution $y(x)$ of (14); such a solution is usually refered to as a *particular solution.* Finding these solutions is usually a matter of trial and error accompanied with educated guess-work — this usually involves looking for a particular solution that is roughly in the same form as the function $f(x)$. To explain here are some examples.

Example 9 *Find the general solution of*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x. \tag{15}$$

As the function on the right is $f(x) = x$ then it would seem sensible to *try* a function of the form

$$Y(x) = Ax + B,$$

where A and B are, as yet, undetermined constants. There is no presumption that such a solution exists, but this seems a sensible range of functions where we may well find a particular solution. Note that

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = A \quad \text{and} \quad \frac{\mathrm{d}^2Y}{\mathrm{d}x^2} = 0.$$

So *if* $Y(x)$ is a solution of (15) then substituting it in gives

$$0 - 3A + 2(Ax + B) = x$$

and this is an equation which must hold for all values of x. So comparing the coefficients of x on both sides, and the constant coefficients, gives

$$\begin{aligned} 2A &= 1 \text{ giving } A = \frac{1}{2}, \\ -3A + 2B &= 0 \text{ giving } B = \frac{3}{4}. \end{aligned}$$

What this means is that

$$Y(x) = \frac{x}{2} + \frac{3}{4}$$

is a particular solution of (15). Having already found the *complementary function* — that is the general solution of the corresponding homogeneous DE in Example (6) then the above proposition tells us that the general solution of (15) is

$$y(x) = Ae^x + Be^{2x} + \frac{x}{2} + \frac{3}{4},$$

for constants A and B.

Example 10 *Find particular solutions of the following DE*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = f(x)$$

where

- $f(x) = \sin x$ — *Simply trying* $Y(x) = A\sin x$ *would do no good as* $Y'(x)$ *would contain* $\cos x$ *terms whilst* $Y(x)$ *and* $Y''(x)$ *would contain* $\sin x$ *terms. Instead we need to try the more general* $Y(x) = A\sin x + B\cos x$;
- $f(x) = e^{3x}$ — *This causes few problems and, as we would expect, we can find a solution of the form* $Y(x) = Ae^{3x}$;
- $f(x) = e^x$ — *This is different to the previous example because we know* Ae^x *is part of the general solution to the corresponding homogeneous DE, and simply substituting in* $Y(x) = Ae^x$ *will yield* 0. *Instead we can successfully try a solution of the form* $Y(x) = Axe^x$.
- $f(x) = xe^{2x}$ — *Again* Ae^{2x} *is part of the solution to the homogeneous DE. Also from the previous example we can see that* Axe^{2x} *would only help us with a* e^{2x} *term on the RHS. So we need to 'move up' a further power and try a solution of the form* $Y(x) = (Ax^2 + Bx)\, e^{2x}$.
- $f(x) = e^x \sin x$ — *Though this may look somewhat more complicated a particular solution of the form* $Y(x) = e^x(A\sin x + B\cos x)$ *can be found.*
- $f(x) = \sin^2 x$ — *Making use of the identity* $\sin^2 x = (1 - \cos 2x)/2$ *we can see that a solution of the form* $Y(x) = A + B\sin 2x + C\cos 2x$ *will work.*

3 Integrating Factors

The method of integrating factors can be used with first order DEs of the form

$$P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)\,y = R(x)\,. \tag{16}$$

The idea behind the method is to rewrite the LHS as the derivative of a product $A(x)\,y$. In general, the LHS of (16) isn't expressible as such, but if we multiply both sides of the DE by an appropriate *integrating factor* $I(x)$ then we can turn the LHS into the derivative of a product.

Let's first of all simplify the equation by dividing through by $P(x)$, and then multiplying by an integrating factor $I(x)$ (which we have yet to determine) to get

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)\frac{Q(x)}{P(x)}y = I(x)\frac{R(x)}{P(x)}. \tag{17}$$

We would like the LHS to be the derivative of a product $A(x)\,y$, which equals

$$A(x)\frac{\mathrm{d}y}{\mathrm{d}x} + A'(x)\,y. \tag{18}$$

So equating the coefficients of y and y' in (17) and (18), we have

$$A(x) = I(x) \quad \text{and } A'(x) = \frac{I(x)\,Q(x)}{P(x)}.$$

Rearranging this gives

$$\frac{I'(x)}{I(x)} = \frac{Q(x)}{P(x)}$$

which has solution

$$I(x) = \exp\int \frac{Q(x)}{P(x)}\,\mathrm{d}x.$$

For such an integrating factor $I(x)$ then (17) now reads as

$$\frac{\mathrm{d}}{\mathrm{d}x}(I(x)\,y) = \frac{I(x)\,R(x)}{P(x)}$$

which has the general solution

$$y(x) = \frac{1}{I(x)}\exp\int \frac{I(x)\,R(x)}{P(x)}\,\mathrm{d}x + \text{const.}$$

Example 11 *Find the general solution of the DE*

$$x\frac{dy}{dx} + (x-1)\,y = x^2.$$

If we divide through by x we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{1}{x}\right)y = x$$

and we see that the integrating factor is

$$\begin{aligned} I(x) &= \exp\int\left(1-\frac{1}{x}\right)\mathrm{d}x \\ &= \exp(x - \log x) \\ &= \frac{1}{x}e^x. \end{aligned}$$

Multiplying through by the integrating factor gives

$$\frac{1}{x}e^{x}\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{1}{x}-\frac{1}{x^{2}}\right)e^{x}y=e^{x},$$

which, by construction rearranges to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}e^{x}y\right)=e^{x}.$$

Integrating gives

$$\frac{1}{x}e^{x}y=e^{x}+K$$

where K is a constant, and rearranging gives

$$y\left(x\right)=x+Kxe^{-x}$$

as our general solution.

Example 12 *Solve the initial value problem*

$$\frac{dy}{dx}+2xy=1,\quad y\left(0\right)=0.$$

The integrating factor here is

$$I\left(x\right)=\exp\int 2x\ \mathrm{d}x=\exp\left(x^{2}\right).$$

Multiplying through we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^{2}}y\right)=e^{x^{2}}\frac{\mathrm{d}y}{\mathrm{d}x}+2xe^{x^{2}}y=e^{x^{2}}.$$

Noting that $y\left(0\right)=0$, when we integrate this we arrive at

$$e^{x^{2}}y=\int_{0}^{x}e^{t^{2}}\ \mathrm{d}t,$$

and rearranging gives

$$y\left(x\right)=e^{-x^{2}}\int_{0}^{x}e^{t^{2}}\ \mathrm{d}t.$$

4 Homogeneous Polar Equations

By a homogeneous polar differential equation we will mean one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right). \tag{19}$$

These can be solved with a substitution of the form

$$y(x) = v(x)\,x \tag{20}$$

to get a new equation in terms of v, x and $\mathrm{d}v/\mathrm{d}x$. Note that the product rule of differentiation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v(x) + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and so making the substitution (20) into the DE (19) gives us the new DE

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + v(x) = f(v),$$

which is a *separable* DE.

Example 13 *Find the general solution of the DE*

$$\frac{dy}{dx} = \frac{x-y}{x+y}.$$

At first glance this may not look like a homogeneous polar DE, but dividing the numerator and denominator in the RHS will quickly dissuade us of this. If we make the substitution $y(x) = xv(x)$ then we have

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x - vx}{x + vx} = \frac{1-v}{1+v}.$$

Rearranging this gives

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1-v}{1+v} - v = \frac{1-2v+v^2}{1+v}$$

and so, separating the variables, we find

$$\int \frac{1+v}{(1-v)^2}\,\mathrm{d}v = \int \frac{\mathrm{d}x}{x}.$$

Using partial fractions gives

$$-\log|1-v| + \frac{2}{1-v} = \int \left(\frac{-1}{1-v} + \frac{2}{(1-v)^2}\right)\mathrm{d}v = \log x + \text{const}.$$

and resubstituting $v = y/x$ leads us to the general solution

$$-\log\left|1 - \frac{y}{x}\right| + \frac{2x}{x-y} = \log x + \text{const}.$$

Example 14 *Solve the initial value problem*

$$\frac{dy}{dx} = \frac{y}{x+y+2} \quad y(0) = 1. \tag{21}$$

This DE is not homogeneous polar, but it can easily be made into such a DE with a suitable change of variables. We introduce new variables

$$X = x + a \quad \text{and} \quad Y = y + b.$$

If we make these substitutions then the RHS becomes

$$\frac{Y - b}{X + Y + 2 - a - b}$$

which is homogeneous if $b = 0$ and $a = 2$. With these values of a and b, noting that

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}\,(y)}{\mathrm{d}\,(x+2)} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

and that the initial condition has become $Y\,(X = 2) = Y\,(x = 0) = y\,(x = 0) = 1$, our initial value problem now reads as

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{Y}{X + Y}, \quad Y\,(2) = 1.$$

Substituting in $Y = VX$ gives us

$$V + X\frac{\mathrm{d}V}{\mathrm{d}X} = \frac{VX}{X + VX} = \frac{V}{1 + V}, \quad V\,(2) = \frac{1}{2}.$$

Rearranging the equation gives us

$$X\frac{\mathrm{d}V}{\mathrm{d}X} = \frac{V}{1 + V} - V = \frac{-V^2}{1 + V},$$

and separating variables gives

$$\frac{1}{V} - \log V = \int \left(-\frac{1}{V^2} - \frac{1}{V} \right) \, \mathrm{d}V = \int \frac{\mathrm{d}X}{X} = \log X + K.$$

Substituting in our initial condition we see

$$2 - \log\left(\frac{1}{2}\right) = \log 2 + K \quad \text{and hence } K = 2.$$

So

$$\frac{1}{V} - \log V = \log X + 2,$$

becomes, when we remember $V = Y/X$,

$$\frac{X}{Y} - \log\left(\frac{Y}{X}\right) = \log X + 2,$$

which simplifies to

$$X - Y \log Y = 2Y.$$

Further, as $X = x + 2$ and $Y = y$, our solution to the initial value problem (21) has become

$$x + 2 = 2y + y \log y.$$