C3.3 Differentiable Manifolds*

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Course overview

A manifold is a space such that small pieces of it look like small pieces of Euclidean space. Thus a smooth surface is an example of a (2-dimensional) manifold. Manifolds are the natural setting for parts of classical applied mathematics such as mechanics, as well as general relativity. They are also central to areas of pure mathematics such as topology and certain aspects of analysis.

In this course we introduce the tools needed to do analysis on manifolds, including vector fields, differential forms and the notion of orientability. We prove a very general form of Stokes' Theorem which includes as special cases the classical theorems of Gauss, Green and Stokes. We also introduce the theory of de Rham cohomology, which is central to many arguments in topology. In particular, we discuss the central notion of degree of smooth maps between manifolds, and its applications. Finally, we briefly discuss Riemannian manifolds, including Riemannian metrics, isometries and geodesics.

Prerequisities. A good understanding of basic topology (e.g. the notions of Hausdorff, open cover, compact, connected etc.) and the theory of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ (including the definition of the differential and the Inverse and Implicit Function Theorems) will be assumed. It will be helpful if you have studied surfaces (both in $\mathbb{R}^3$ and "abstract" surfaces), but not essential.

Disclaimer. These lecture notes cover the essential course material (and a bit more), but there are no pictures and possibly a few typos. The lectures will provide additional motivation and intuition which will be invaluable for understanding and appreciating the material. Moreover, I would suggest combining these lecture notes with material from the recommended reading below.

Recommended texts

- D. Barden and C. Thomas, An Introduction to Differential Manifolds. (Imperial College Press, London, 2003.)
- M. Berger and B. Gostiaux, Differential Geometry: Manifolds, Curves and Surfaces. Translated from the French by S. Levy, (Springer Graduate Texts in Mathematics, 115, Springer–Verlag (1988)) Chapters 0-3, 5-7.
- W. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd edition, (Academic Press, 1986).
- M. Spivak, Calculus on Manifolds, (W. A. Benjamin, 1965).
- M. Spivak, A Comprehensive Introduction to Differential Geometry, Vol. 1, (1970).
- F. Warner, Foundations of Differentiable Manifolds and Lie Groups, (Springer Graduate Texts in Mathematics, 1994).

The best books for the course are probably Barden and Thomas, Boothby and Spivak (Calculus on Manifolds).

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Contents

1 Manifolds: definition and examples

We want to start by talking about one of the key building blocks of modern geometry: manifolds. All of the manifolds we will discuss, as the name of this course suggests, will be *differentiable* manifolds, and so we will omit the adjective "differentiable" for simplicity (as is standard practice in this area).

For the moment, let us make a fake definition of manifold and see some examples.

First fake definition: A manifold is the natural notion of a smooth object.

Although this definition is fake, it is useful in the sense that everything that you would imagine to be a smooth object (and thus a manifold) is a manifold. Moreover, the actual definition is not very enlightening. We need it, so that all of the theory makes sense, but once we have it we then very rarely need to use it.

1.1 Basic examples

Example. $\mathbb{R}^2$ is a 2-dimensional manifold and in general $\mathbb{R}^n$ is an n-dimensional manifold.

Example. The upper-half plane

$$H^2 = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > 0\}$$

is a 2-dimensional manifold. Similarly, the n-dimensional upper half-space

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_n > 0\}$$

is an n-dimensional manifold.

Example. The unit disk

$$B^2 = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + x_2^2 < 1\}$$

is a 2-dimensional manifold. Similarly, the unit ball in $\mathbb{R}^n$,

$$B^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, |x|^2 = \sum_{i=1}^n x_i^2 < 1\}$$

is an n-dimensional manifold.

Example. The n-dimensional sphere

$$\mathcal{S}^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,:\, |x|^2 = \sum_{i=1}^{n+1} x_i^2 = 1\}$$

is an n-dimensional manifold.

Example. The torus

$$\{((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta) \in \mathbb{R}^3 : \theta, \phi \in \mathbb{R}\}$$

in $\mathbb{R}^3$ is a 2-dimensional manifold.

Example. The n-dimensional torus $T^n \subseteq \mathbb{R}^{2n}$ given by

$$T^n = \{(\cos\theta_1, \sin\theta_1, \ldots, \cos\theta_n, \sin\theta_n) \in \mathbb{R}^{2n}\,;\, \theta_1, \ldots, \theta_n \in \mathbb{R}\}$$

is an n-dimensional manifold.

The previous two examples give two possible realisations of the 2-dimensional torus: either in $\mathbb{R}^3$ or in $\mathbb{R}^4$. Are these the same? If not, how are they different?

1.2 Some non-examples

So what is a manifold? The simplest example of an n-dimensional manifold that we have seen is just $\mathbb{R}^n$, and this is the local model for all manifolds.

Second fake definition: An n-dimensional manifold is something which locally "looks like" $\mathbb{R}^n$ (but globally can be much more interesting).

For example, if you take a sphere in $\mathbb{R}^3$, it is clearly not just flat $\mathbb{R}^2$, but if you look near any given point you can define coordinates so it looks like a piece of $\mathbb{R}^2$. The same trick can be done for all of the examples we have seen so far. With this second fake definition we may ask the question: what is not a manifold?

Example. A cube is not a manifold. It is not smooth at the edges and at the corners. Indeed, it looks like $\mathbb{R}^2$ on the faces, but not at the edges or at the corners.

Similarly, any polyhedron is not a manifold.

Example. The closed disk in $\mathbb{R}^2$

$$\{x \in \mathbb{R}^2 \,:\, |x| \leq 1\}$$

is not quite a 2-dimensional manifold because it looks like $\mathbb{R}^2$ in the interior where $|x| < 1$, but when $|x| = 1$ we have the circle $\mathcal{S}^1$. (However, it is what is called a 2-dimensional manifold with boundary.)

Example. The hyperboloid of one sheet

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 - x_3^2 = 1\}$$

and the hyperboloid of two sheets

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 - x_3^2 = -1\}$$

are 2-dimensional manifolds, but

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 - x_3^2 = 0\}$$

is a cone and so is not a manifold, because it is not smooth at 0, or it does not look like $\mathbb{R}^2$ there.

1.3 More advanced examples

Now, everything we have looked at so far has been quite concrete, but one of the great powers of the theory of manifolds is that they include much more abstract objects. Let us now look at two more abstract things, which I claim are manifolds, and we shall see why shortly.

Example. Let $M_n(\mathbb{R})$ be the $n \times n$ real matrices. Then the general linear group is

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \,:\, \det A \neq 0\}$$

and the special linear group is

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \,:\, \det A = 1\}.$$

(Notice that these are groups under multiplication because the identity matrix is in both, and $\det(AB) = \det A \det B$.) Then $\mathrm{GL}(n, \mathbb{R})$ is an n^2-dimensional manifold and $\mathrm{SL}(n, \mathbb{R})$ is an $n^2 - 1$-dimensional manifold.

Example. Let I be the identity matrix in $M_n(\mathbb{R})$. Then

$$\mathrm{O}(n) = \{A \in M_n(\mathbb{R}) \,:\, A^{\mathrm{T}} A = I\} \quad \text{and} \quad \mathrm{SO}(n) = \{A \in \mathrm{O}(n) \,:\, \det(A) = 1\}$$

are the orthogonal and special orthogonal group, respectively. (Again, notice that these are groups under multiplication because I is in both and $(AB)^{\mathrm{T}}(AB) = B^{\mathrm{T}}A^{\mathrm{T}}AB$.) Then $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are $\frac{1}{2}n(n-1)$-dimensional manifolds.

Example. Let

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,:\, a, b \in \mathbb{C},\, |a|^2 + |b|^2 = 1 \right\}.$$

This is again a group and is a 3-dimensional manifold. In general, if we let $M_n(\mathbb{C})$ be the $n \times n$ complex matrices, then the special unitary group

$$\mathrm{SU}(n) = \left\{ A \in M_n(\mathbb{C}) \,:\, \overline{A^{\mathrm{T}}}A = I,\, \det A = 1 \right\}$$

is an $n^2 - 1$-dimensional manifold and the unitary group

$$\mathrm{U}(n) = \left\{ A \in M_n(\mathbb{C}) \,:\, \overline{A^{\mathrm{T}}}A = I \right\}$$

is an n^2-dimensional manifold. (These are, sort of, complex analogues of the special orthogonal and orthogonal groups.)

Remark. The examples just given in terms of matrices are all examples of manifolds which are groups: in fact, this is almost the definition of a Lie group (we will see the correct definition later), and these examples are indeed all Lie groups.

We can even go a bit more abstract, and produce some more interesting spaces which play an important role in the study of manifolds.

Example. Let $\mathbb{RP}^n$ be the set of straight lines in $\mathbb{R}^{n+1}$ through 0. Then $\mathbb{RP}^n$ is the real projective n-space and is an n-dimensional manifold.

We can equivalently say that $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R}$. Hence, we usually denote points in $\mathbb{RP}^n$ (which represent lines in $\mathbb{R}^{n+1}$) by equivalence classes $[x]$ (where $x \in \mathbb{R}^{n+1} \setminus \{0\}$ lies on the line).

Example. We have that $\mathbb{C}^n$ is a $2n$-dimensional manifold. We can then consider the set $\mathbb{CP}^n$ of complex lines in $\mathbb{C}^{n+1}$ through 0. This is also a $2n$-dimensional manifold, called complex projective n-space.

More explicitly, $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation $z \sim w$ if $z = \lambda w$ for some $\lambda \in \mathbb{C}$. Again, we tend to denote points in $\mathbb{CP}^n$ by equivalence classes $[z]$.

1.4 Constructing manifolds: regular values

Let us put off the formal definition of manifold a little bit longer and give a general technique for constructing manifolds which is very helpful.

Recall: If we write $F : \mathbb{R}^n \to \mathbb{R}^m$ as $F(x) = (F_1(x), \ldots, F_m(x))$ then the derivative of F at p is the linear map $\mathrm{d}F_p : \mathbb{R}^n \to \mathbb{R}^m$ which is represented by the matrix $(\frac{\partial F_i}{\partial x_j}(p))$. In general, $\mathrm{d}F_p$ is the linear map so that

$$\frac{|F(p+h) - F(p) - \mathrm{d}F_p(h)|}{|h|} \to 0 \quad \text{as } |h| \to 0.$$

Remark. We will say that a map is smooth if it is infinitely differentiable (i.e. C^∞). Many statements we make in this course can naturally be generalised to the case where maps have weaker differentiability properties, but in this course we will restrict ourselves to the smooth category.

Theorem 1.1. (Regular value theorem). *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth map and suppose that for all $p \in F^{-1}(c)$, where*

$$F^{-1}(c) = \{p \in \mathbb{R}^{n+m} : F(p) = c\} \neq \emptyset,$$

the derivative $\mathrm{d}F_p : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is surjective (i.e. c is a regular value of F). Then $F^{-1}(c)$ is an n-dimensional manifold.

Remark. This would obviously work just as well if F is only defined on an open set in $\mathbb{R}^{n+m}$.

Let us put this theorem to use straight away.

Example. Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be

$$F(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2.$$

Notice that F is a smooth map (since it is polynomial) with

$$\mathrm{d}F_x = (2x_1 \ \dots \ 2x_{n+1}).$$

If $x \in F^{-1}(1)$ then $\mathrm{d}F_x \neq 0$, but if $x \in F^{-1}(0)$ then $\mathrm{d}F_x = 0$. In this case, the derivative being non-zero is equivalent to saying that the map $\mathrm{d}F_x : \mathbb{R}^{n+1} \to \mathbb{R}$ is surjective, so 1 is a regular value but 0 is not.

Hence, by Theorem 1.1, we see that $F^{-1}(1) = \mathcal{S}^n$ is an $(n+1) - 1 = n$-dimensional manifold. We also notice that $F^{-1}(0) = \{0\}$, which is obviously not an n-dimensional manifold.

Example. Let $F : \mathbb{R}^{2n} \to \mathbb{R}^n$ be

$$F(x_1, \dots, x_{2n}) = (x_1^2 + x_2^2 - 1, \dots, x_{2n-1}^2 + x_{2n}^2 - 1).$$

This is a smooth map and for $x \in F^{-1}(0)$,

$$\mathrm{d}F_x = \begin{pmatrix} 2x_1 & 2x_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2x_3 & 2x_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2x_{2n-1} & 2x_{2n} \end{pmatrix}$$

has rank n as a matrix because the rows are all linearly independent, as $(x_{2i-1}, x_{2i}) \neq (0,0)$ for all i. We deduce that $\mathrm{d}F_x$ is a surjective map. Thus $F^{-1}(0) = T^n \subseteq \mathbb{R}^{2n}$ is an n-dimensional manifold by Theorem 1.1.

Our next example shows that, although being a regular value of a function is a sufficient condition to ensure that the level set is a manifold, it is not necessary.

Example. Let $F(x_1, x_2) = x_1^3 - x_2^3$. Then $\mathrm{d}F_{(x_1,x_2)} = (3x_1^2 \ \ -3x_2^2)$ so 0 is not a regular value of F because $\mathrm{d}F_{(0,0)} = 0$. However

$$F^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^3 = x_2^3\} = \{(x_1, x_1) \in \mathbb{R}^2 : x_1 \in \mathbb{R}\}$$

which is a 1-dimensional manifold (just a diagonal line in the plane).

We now give a more sophisticated implementation of Theorem 1.1.

Example. Define $F : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $F(A) = A^{\mathrm{T}}A - I$. Notice that $F(A)^{\mathrm{T}} = F(A)$, so F actually maps into $\mathrm{Sym}_n(\mathbb{R})$, the symmetric $n \times n$ matrices. Notice that $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $\mathrm{Sym}_n(\mathbb{R}) \cong \mathbb{R}^{\frac{1}{2}n(n+1)}$. Clearly F is smooth and we compute its derivative, which is the part of $F(A+B) - F(A)$ which is linear in B. Explicitly, we see that

$$F(A+B) - F(A) = (A+B)^{\mathrm{T}}(A+B) - A^{\mathrm{T}}A = B^{\mathrm{T}}A + A^{\mathrm{T}}B + B^{\mathrm{T}}B.$$

Hence,

$$\frac{|F(A+B)-F(A)-(B^{\mathrm{T}}A+A^{\mathrm{T}}B)|}{|B|}=\frac{|B^{\mathrm{T}}B|}{|B|}\to 0$$

as $|B|\to 0$, so

$$\mathrm{d}F_A(B)=B^{\mathrm{T}}A+A^{\mathrm{T}}B.$$

If $C\in \mathrm{Sym}_n(\mathbb{R})$ and $A\in F^{-1}(0)$ then, since $C^{\mathrm{T}}=C$ and $A^{\mathrm{T}}A=I$, we have

$$\mathrm{d}F_A(\frac{1}{2}AC)=\frac{1}{2}(AC)^{\mathrm{T}}A+\frac{1}{2}A^{\mathrm{T}}AC=\frac{1}{2}C^{\mathrm{T}}A^{\mathrm{T}}A+\frac{1}{2}C=C,$$

so $\mathrm{d}F_A$ is surjective. Applying Theorem 1.1 gives that $\mathrm{O}(n)=F^{-1}(0)$ is an $n^2-\frac{1}{2}n(n+1)=\frac{1}{2}n(n-1)$-dimensional manifold.

Notice that $\mathrm{O}(n)$ is the disjoint union of two manifolds because for all $A\in\mathrm{O}(n)$, $\det(A)=\pm1$, so $\mathrm{SO}(n)$ is an open subset of $\mathrm{O}(n)$, and hence is also an $\frac{1}{2}n(n-1)$-dimensional manifold. You can also show this directly using the same argument as above replacing $M_n(\mathbb{R})$ by the open subset $\mathrm{GL}^+(n,\mathbb{R})=\{A\in M_n(\mathbb{R})\,:\,\det A>0\}$.

Remark. Theorem 1.1 is extremely helpful. For example, it says that, most of the time, if you look at the zero set of a system of polynomials you will get a manifold if the system has a root. This is particularly helpful in complex Euclidean space (or complex projective space, as long as you use homogeneous polynomials), since complex polynomials always have roots (by the Fundamental Theorem of Algebra). Zero sets of systems of polynomials in complex projective space are important objects in algebraic geometry.

1.5 The formal definition

The formal definition of a manifold may look a bit strange but the key points we want are:

- "abstract" objects – e.g. the usual torus in $\mathbb{R}^3$ we know should be "the same" as $T^2=\mathcal{S}^1\times\mathcal{S}^1\subseteq\mathbb{R}^4$;
- smooth geometric objects – e.g. we want things like the sphere but we want to rule out the cube and the cone;
- objects on which we can measure how quantities vary as we move from point to point – i.e. we can define differentation.

With these properties, the definition of manifold becomes essentially determined.

Definition 1.2. An n-dimensional *manifold* is a (second countable, Hausdorff) topological space M such that there exists a family $\mathcal{A}=\{(U_i,\varphi_i)\,:\,i\in I\}$ where:

- U_i is an open set in M and $\cup_{i\in I}U_i=M$;
- $\varphi_i\,:\,U_i\to\mathbb{R}^n$ is a continuous bijection onto an open set $\varphi_i(U_i)$ with continuous inverse (i.e. a homeomorphism);
- whenever $U_i\cap U_j\neq\emptyset$, the *transition map* $\varphi_j\circ\varphi_i^{-1}:\varphi_i(U_i\cap U_j)\to\varphi_j(U_i\cap U_j)$ is a smooth (infinitely differentiable) bijection with smooth inverse (i.e. a diffeomorphism).

The family $\mathcal{A}$ is called an *atlas* and the pairs (U_i,φ_i) are called (coordinate) *charts*. (We are pasting together little flat maps to get a picture of a non-flat object, just like with the Earth.)

Remark. (Not examinable). Second countable means there is a countable collection of open sets which form a basis for all open sets: for $\mathbb{R}^n$, for example, we can take the set of balls $B_r(x)$ with centre $x\in\mathbb{Q}^n$ and radius $r\in\mathbb{Q}^+$. Recall that Hausdorff means that for any distinct points x and y we can find disjoint open sets containing x and y. Almost every example of a manifold in this course will come from a metric space, where second countable is the same thing as separable. Separable means there is a countable dense subset: for $\mathbb{R}^n$ just take $\mathbb{Q}^n$ as the countable dense subset (dense just means its closure is the whole space, or every non-empty open set meets the dense set). Since our emphasis is on geometry, we will not worry about these sort of topological issues in this course.

Remark. (Not examinable). One can also weaken the definition of manifold by requiring less differentiability for the transition maps, or by relaxing the topological assumptions (e.g. dropping the second countable requirement), but this will mean that certain constructions that arise in this course are not possible or need to be adapted.

As we already said, most of the time we do not need the definition of manifold: we just need to know that something is one. As we have seen, everything you would like to be a manifold is one. However, it is instructive to check that some things really fit the definition of manifold.

Let us now try to give some examples of manifolds from the definition.

Example. $\mathbb{R}^n$ is an n-dimensional manifold: take $U = \mathbb{R}^n$, $\varphi = \mathrm{id}$ (the identity map) and $\mathcal{A} = \{(U, \varphi)\}$. The same works for any open set in $\mathbb{R}^n$, so this shows that H^n and B^n are n-dimensional manifolds.

Example. In fact, any open subset U of a manifold M is a manifold of the same dimension – take the atlas $\{(U_i \cap U, \varphi_i|_{U_i \cap U}) : i \in I\}$ if $\{(U_i, \varphi_i) : i \in I\}$ is an atlas for M.

In particular since $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ (we have n^2 independent real entries in the matrix) and $\mathrm{GL}(n, \mathbb{R})$ is an open set in $M_n(\mathbb{R})$ (since the condition $\det A \neq 0$ is an open condition), we have that $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}^+(n, \mathbb{R})$ are n^2-dimensional manifolds.

Example. Consider $\mathcal{S}^n$.

- Let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$ be the "North" and "South" poles. Let $U_N = \mathcal{S}^n \setminus \{N\}$ and $U_S = \mathcal{S}^n \setminus \{S\}$. These are open sets and $U_N \cup U_S = \mathcal{S}^n$.
- Let $\varphi_N : U_N \to \mathbb{R}^n$ be given by
$$\varphi_N(x) = \frac{(x_1, \ldots, x_n)}{1 - x_{n+1}}$$
and $\varphi_S : U_S \to \mathbb{R}^n$ be given by
$$\varphi_S(x) = \frac{(x_1, \ldots, x_n)}{1 + x_{n+1}}.$$
(These are the stereographic projections.) We have explicit inverses (if we write $|y|^2 = \sum_{i=1}^n y_i^2$):
$$\varphi_N^{-1}(y) = \left(\frac{2y_1}{1 + |y|^2}, \ldots, \frac{2y_n}{1 + |y|^2}, \frac{|y|^2 - 1}{1 + |y|^2} \right)$$
and
$$\varphi_S^{-1}(y) = \left(\frac{2y_1}{1 + |y|^2}, \ldots, \frac{2y_n}{1 + |y|^2}, \frac{1 - |y|^2}{1 + |y|^2} \right)$$
so φ_N, φ_S are clearly homeomorphisms.
- $U_N \cap U_S = \mathcal{S}^n \setminus \{N, S\}$, $\varphi_N(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$ and $\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is
$$\varphi_S \circ \varphi_N^{-1}(y) = \frac{y}{|y|^2},$$
which is a diffeomorphism because it is smooth, as $y \neq 0$, and it is its own inverse. (Essentially the transition map is the "inversion" map.)

So, the conditions of Definition 1.2 are satisfied and $\mathcal{S}^n$ is an n-dimensional manifold.

Example. For $\mathbb{RP}^n$ we have the following atlas.

- For $i = 1, \ldots, n+1$ we let $U_i = \{[(x_1, \ldots, x_{n+1})] \in \mathbb{RP}^n : x_i \neq 0\}$.
- We define $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([x]) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

Then the conditions of Definition 1.2 are satisfied for $\{(U_i, \varphi_i) : i = 1, \ldots, n+1\}$ and $\mathbb{RP}^n$ is an n-dimensional manifold.

Example. Similarly, for $\mathbb{CP}^n$ we have the following atlas.

- For $i = 1, \ldots, n+1$ we let $U_i = \{[(z_1, \ldots, z_{n+1})] \in \mathbb{CP}^n : z_i \neq 0\}$.
- We define $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i([z]) = \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{n+1}}{z_i}\right).$$

Then the conditions of Definition 1.2 are satisfied for $\{(U_i, \varphi_i) : i = 1, \ldots, n+1\}$ and $\mathbb{CP}^n$ is a $2n$-dimensional manifold.

Remark. (Not examinable.) In fact, the previous example shows that $\mathbb{CP}^n$ is an n-dimensional *complex manifold.*

Remark. (Not examinable). In these examples, really what we have shown is that they have atlases which give them a manifold structure. We could have chosen different atlases which could give different manifold structures. However, if atlases are *equivalent* they give the same manifold structure, and an equivalence class of atlases is called a smooth structure. Two atlases $\{(U_i, \varphi_i) : i \in I\}$ and $\{(V_j, \psi_j) : j \in J\}$ are equivalent if whenever $U_i \cap V_j \neq \emptyset$ the maps $\psi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap V_j) \to \psi_j(U_i \cap V_j)$ and $\varphi_i \circ \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \varphi_i(U_i \cap V_j)$ are diffeomorphisms, which is the same as saying that the union of the two atlases is still an atlas. Therefore to define a manifold with a smooth structure one only needs one atlas, as in Definition 1.2.

Remark. (Not examinable). One can take an alternative approach to the definition of a manifold which bypasses the need for an underlying choice of topology. One can just assume that M in Definition 1.2 is a set and we can define an atlas $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ by the conditions:

- U_i is a subset of M for each i with $\cup_{i \in I} U_i = M$;
- $\varphi_i : U_i \to \mathbb{R}^n$ is a bijection onto an open set $\varphi_i(U_i)$;
- for all $i, j \in I$, $\varphi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$;
- whenever $U_i \cap U_j \neq \emptyset$, the transition map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a smooth bijection with smooth inverse.

Using the same notion of equivalence of atlases and smooth structure discussed in the previous remark, we can then define a manifold to be a set M endowed with a smooth structure. We may also define a topology on M from the smooth structure by demanding that V is open in M if and only if $\varphi_i(V \cap U_i)$ is open in $\mathbb{R}^n$ for all $i \in I$. It is clear this definition is independent of the choice of atlas in the equivalence class given by the smooth structure. This is convenient in that we just need a set with an atlas to define a manifold. However, one then has to impose that the induced topology is second countable and Hausdorff by hand in order to proceed further.

As you see, proving that something is a manifold from the definition is quite laborious so we will try to avoid it!

For completeness, I give a sketch proof of the regular value theorem for constructing manifolds from the definition.

Proof of Theorem 1.1. **(Not examinable).**

- Applying the Implicit Function Theorem shows that for all $p \in F^{-1}(c)$ there exists a splitting of $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m = \operatorname{Ker} \mathrm{d}F_p \times \mathbb{R}^m$ such that, if $p = (a, b)$ with respect to this splitting, then there exist open sets $a \in V_p \subseteq \mathbb{R}^n$ and $b \in W_p \subseteq \mathbb{R}^m$ and a smooth map $G_p : V_p \to W_p$ with $G_p(a) = b$ such that
$$F^{-1}(c) \cap (V_p \times W_p) = \{(q, G_p(q)) \,:\, q \in V_p\}.$$
Let $U_p = F^{-1}(0) \cap (V_p \times W_p)$ which is an open set and $\cup_{p\in F^{-1}(0)} U_p = F^{-1}(0)$ (since $p \in U_p$).

- For all $p \in F^{-1}(0)$ let $\varphi_p : U_p \to V_p \subseteq \mathbb{R}^n$ be given by $\varphi_p\big(q, G_p(q)\big) = q$. Then $\varphi_p^{-1}(q) = (q, G_p(q))$ so it is a homeomorphism.

- Claim: the transition maps $\varphi_p \circ \varphi_{p'}^{-1}$ are smooth.

Hence $F^{-1}(c)$ satisfies the conditions of Definition 1.2 and is an n-dimensional manifold. □

1.6 Smooth maps

Now that we have some manifolds and seen that many natural objects are manifolds, we want to see why they are useful. We begin with the idea that we can measure how quantities vary on manifolds; i.e. that we can differentiate. This really shows why the manifold definition is what it is. The point is that we can differentiate maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, so we use the definition we know there.

Definition 1.3. Let M and N be manifolds of dimensions m and n respectively. A map $f : M \to N$ is *smooth* at p if for some coordinate charts (U, φ) at p and (V, ψ) at $f(p)$ with $f(U) \subseteq V$, the map
$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$$
is smooth. We say f is smooth if it is smooth at all $p \in M$.

Remark. This definition makes sense precisely because of Definition 1.2: if we take $(U, \varphi), (U', \varphi')$ around p and $(V, \psi), (V', \psi')$ around $f(p)$ with $f(U') \subseteq V'$ and $f(U) \subseteq V$ then
$$\psi' \circ f \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$
so the left-hand side is smooth if and only if $\psi \circ f \circ \varphi^{-1}$ is smooth because the transition maps are smooth.

Basically every map that we care about between manifolds will be smooth and we won't need to check it, but just for completeness here are a couple of examples.

Example. The maps φ in the atlas for an n-dimensional manifold M are smooth maps from M to $\mathbb{R}^n$ since we can take (U, φ) as a coordinate chart in M and $(\mathbb{R}^n, \mathrm{id})$ as the chart for $\mathbb{R}^n$ and $\mathrm{id} \circ \varphi \circ \varphi^{-1} = \mathrm{id} : \varphi(U) \to \varphi(U)$ is smooth.

The maps φ^{-1} are also smooth in a similar way since $\varphi \circ \varphi^{-1} \circ \mathrm{id} = \mathrm{id} : \varphi(U) \to \varphi(U)$ is smooth.

Example. The identity map $\mathrm{id} : M \to M$ is smooth because given any chart (U, φ) on M we have that $\varphi \circ \mathrm{id} \circ \varphi^{-1} = \mathrm{id}$ on $\varphi(U)$, which is smooth.

Example. If $M \subseteq \mathbb{R}^n$ is a manifold, then the restriction of any smooth map on $\mathbb{R}^n$ to M is a smooth map.

If $N \subseteq \mathbb{R}^m$ is also a manifold and the map $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth such that $f(M) \subseteq N$ then the restriction $f : M \to N$ is smooth.

These are the cases we will mainly be interested in. For example, if we take the map $f : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$f(x_0, x_1, x_2, x_3) = \left(x_0^2 + x_1^2 - x_2^2 - x_3^2, 2x_0x_3 + 2x_1x_2, 2x_1x_3 - 2x_0x_2\right),$$

we see that it is smooth and if $(x_0, x_1, x_2, x_3) \in \mathcal{S}^3$ then $f(x_0, x_1, x_2, x_3) \in \mathcal{S}^2$. Hence, its restriction is a smooth map $f : \mathcal{S}^3 \to \mathcal{S}^2$.

Example. For any of the groups G of matrices we have discussed, the multiplication map $m : \mathrm{G} \times \mathrm{G} \to \mathrm{G}$ given by $m(A, B) = AB$ and the inversion map $i : \mathrm{G} \to \mathrm{G}$ given by $i(A) = A^{-1}$ are smooth. This is what makes them Lie groups.

The left and right multiplication maps $L_A : \mathrm{G} \to \mathrm{G}$ and $R_A : \mathrm{G} \to \mathrm{G}$ given by $L_A(B) = AB$ and $R_A(B) = BA$ are smooth. Moreover, the determinant $\det : \mathrm{G} \to \mathbb{R}$ and trace $\mathrm{tr} : \mathrm{G} \to \mathbb{R}$ are smooth.

We will be interested mainly in special types of smooth map which will help us relate two different manifolds.

Definition 1.4. A map $f : M \to N$ is a *diffeomorphism* if it is a smooth bijection with a smooth inverse. The manifolds M and N are then said to be diffeomorphic.

A map $f : M \to N$ is a *local diffeomorphism* at p if $\exists$ open $U \ni p$, open $V \ni f(p)$ such that $f : U \to V$ is a diffeomorphism. We say f is a local diffeomorphism if it is a local diffeomorphism at all $p \in M$.

A diffeomorphism is the natural notion of equivalence between manifolds, so diffeomorphic manifolds are "the same".

Example. The identity map $\mathrm{id} : M \to M$ is a diffeomorphism.

If f, g are diffeomorphisms then so is $f \circ g$ and so is f^{-1}.

Hence, the diffeomorphisms form a group which we write $\mathrm{Diff}(M)$.

Example. The maps $\varphi : U \to \mathbb{R}^n$ in the atlas for an n-dimensional manifold M are local diffeomorphisms because the maps $\varphi : U \to \varphi(U)$ are diffeomorphisms by Definition 1.2. This justifies the statement that manifolds always locally "look like" $\mathbb{R}^n$.

Example. Any matrix $A \in M_n(\mathbb{R})$ defines a linear map on $\mathbb{R}^n$ given by $x \mapsto Ax$ which is smooth, since any linear map is smooth. Moreover, this map is invertible if and only if A is invertible with inverse $x \mapsto A^{-1}x$, which is if and only if $A \in \mathrm{GL}(n, \mathbb{R})$. Since the inverse is also linear, it is smooth, so $A \in \mathrm{GL}(n, \mathbb{R})$ defines a diffeomorphism on $\mathbb{R}^n$.

We thus see that the group of linear diffeomorphisms of $\mathbb{R}^n$ is $\mathrm{GL}(n, \mathbb{R})$.

Example. The map $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $f(x) = \tan x$ is smooth and its inverse $f^{-1} = \tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is smooth so f is a diffeomorphism.

This example shows that any open interval in $\mathbb{R}$ is diffeomorphic to $\mathbb{R}$, just by rescaling and translating the endpoints of the interval.

Example. The left and right multiplication maps L_A, R_A are diffeomorphisms on the groups G.

Example. The map $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}_{>0}$ given by $f(x) = x^2$ is a local diffeomorphism. It is smooth and surjective and obviously not injective since $f(-x) = f(x)$. However, the restricted maps $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and $f : \mathbb{R}_{<0} \to \mathbb{R}_{>0}$ are diffeomorphisms so f is a local diffeomorphism.

In fact, there will be no diffeomorphism from $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}_{>0}$ because in fact there is no homeomorphism between them (one is connected while the other is disconnected).

1.7 Quotient constructions

We can now give another very useful way to construct manifolds. For this we need group actions.

Definition 1.5. We say that a group G acts on M by diffeomorphisms if there is a homomorphism $G \to \mathrm{Diff}(M)$; i.e. for all $g \in G$ there exists a diffeomorphism f_g of M so that

- $f_e = \mathrm{id}$ (where e is the identity in G);
- $f_{gh} = f_g \circ f_h$ $\forall g, h \in G$.

Let G be a discrete group (i.e. a finite group or $\mathbb{Z}^n$ or some other countable group) acting on M by diffeomorphisms as above. We say that G acts freely and properly discontinuously if

- $\forall p \in M$ $\exists$ open $V \ni p$ with $V \cap f_g(V) = \emptyset$ $\forall g \neq e$;
- $\forall p, q \in M$ with $p \neq f_g(q)$ $\forall g \in G$ $\exists$ open $V \ni p$ and $W \ni q$ with $V \cap f_g(W) = \emptyset$ $\forall g \in G$.

Notice that the first one essentially says that f_g has no fixed points if $g \neq e$.

Theorem 1.6. *Let M be an n-dimensional manifold and let G be a discrete group acting freely and properly discontinuously on M by diffeomorphisms. Define an equivalence relation $\sim$ on M by $p \sim q \Leftrightarrow q = f_g(p)$ for some $g \in G$. Then the quotient space $M/\sim = M/G$ is an n-dimensional manifold.*

Proof. **(Not examinable).**

- Let $\{(V_i, \psi_i) : i \in I\}$ be an atlas for M such that $V_i \cap f_g(V_i) = \emptyset$ $\forall g \neq e$. (This is possible by the definition of a properly discontinuous action, because for all p we can find $V \ni p$ such that $V \cap f_g(V) = \emptyset$ for all $g \neq e$). Let $\pi : M \to M/G$ be the projection map which is an open map. Then $U_i = \pi(V_i)$ is open and $\cup_{i\in I} U_i = M/G$.
- Since $\pi_i = \pi|_{V_i} : V_i \to U_i$ is a homeomorphism (it is injective because $V_i \cap f_g(V_i) = \emptyset$ for $g \neq e$), we can define $\varphi_i = \psi_i \circ \pi_i^{-1} : U_i \to \psi_i(V_i) \subseteq \mathbb{R}^n$ which is a homeomorphism.
- If $U_i \cap U_j \neq \emptyset$ then
$$\varphi_i(U_i \cap U_j) = \psi_i \circ \pi_i^{-1}(U_i \cap U_j) = \psi_i(V_i \cap \pi^{-1}(U_j)) = \psi_i\big(V_i \cap \cup_{g\in G} f_g(V_j)\big),$$
which is a disjoint union of open sets and clearly $\varphi_j \circ \varphi_i^{-1}$ is a homeomorphism, so it is enough to show that $\varphi_j \circ \varphi_i^{-1}$ (and its inverse) are smooth.

  Let $p \in \varphi_i(U_i \cap U_j)$. Then there exists unique $g \in G$ such that $p \in W = \psi_i(V_i \cap f_g(V_j))$. Then $\psi_i^{-1}(W) = V_i \cap f_g(V_j)$ and
$$\varphi_j \circ \varphi_i^{-1}|_W = \psi_j \circ \pi_j^{-1} \circ \pi_i \circ \psi_i^{-1}|_W$$
so it is enough to show that $\pi_j^{-1} \circ \pi_i$ is smooth on $V = V_i \cap f_g(V_j)$. If $q \in V$ and $q' = \pi_j^{-1} \circ \pi_i(q) \in V_j$ then $\pi_j(q') = \pi_i(q)$ so there exists $g_q \in G$ such that $f_{g_q}(q') = q$. Therefore $q \in f_{g_q}(V_j) \cap f_g(V_j)$ so $g_q = g$ and hence $\pi_j^{-1} \circ \pi_i|_V = f_{g^{-1}}|_V$ which is smooth. Hence $\varphi_j \circ \varphi_i^{-1}$ is smooth, and the same argument clearly works for the inverse.

We see that the conditions of Definition 1.2 are satisfied. □

Theorem 1.6 gives an invaluable tool for constructing examples of manifolds.

Examples. Let $\mathbb{Z}_2 = \{-1, 1\}$ act on $\mathbb{R}^n$ with $f_1 = \mathrm{id}$ and $f_{-1} = -\mathrm{id}$. Clearly $-\mathrm{id}$ is a diffeomorphism of $\mathbb{R}^n$ but it is not a free action because 0 is fixed. However, it is pretty clear that $\mathbb{Z}_2$ acts freely and properly discontinuously by diffeomorphisms on $\mathbb{R}^n \setminus \{0\}$. For completeness, we show explicitly that this is true.

If we take any point $x = (x_1, \ldots, x_n) \neq 0$ in $\mathbb{R}^n$ then there exists some coordinate $x_i \neq 0$. Let $V \ni x$ be the open set

$$V = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \, : \, |y_i - x_i| < |x_i|\}.$$

Then if $y \in V \cap (-V)$ we must have

$$|x_i| = |\frac{1}{2}(x_i - y_i) + \frac{1}{2}(y_i + x_i)| \leq \frac{1}{2}|x_i - y_i| + \frac{1}{2}|x_i + y_i| < \frac{1}{2}|x_i| + \frac{1}{2}|x_i| = |x_i|,$$

which is a contradiction. Hence $V \cap (-V) = V \cap f_{-1}(V) = \emptyset$.

Similarly, if $x, y \in \mathbb{R}^n$ with $y \neq x$ and $y \neq -x$ then there exists some coordinates so that $y_i \neq x_i$ and $y_j \neq -x_j$. So, we take

$$V = \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n \, : \, |z_i - x_i| < \frac{|x_i - y_i|}{2}, \, |z_j - x_j| < \frac{|x_j + y_j|}{2}\}$$

and

$$W = \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n \, : \, |z_i - y_i| < \frac{|x_i - y_i|}{2}, \, |z_j - y_j| < \frac{|x_j + y_j|}{2}\}$$

so that V, W are open and $x \in V$ and $y \in W$. We also see that if $z \in V \cap W$ we would have

$$|x_i - y_i| = |x_i - z_i + z_i - y_i| \leq |x_i - z_i| + |z_i - y_i| < \frac{|x_i - y_i|}{2} + \frac{|x_i - y_i|}{2} = |x_i - y_i|,$$

which is a contradiction so $V \cap W = V \cap f_1(W) = \emptyset$. Similarly, if $z \in V \cap (-W)$ (which means $-z \in W$) then

$$|x_j + y_j| = |x_j - z_j + z_j + y_j| \leq |x_j - z_j| + |z_j + y_j| < \frac{|x_j + y_j|}{2} + \frac{|x_j + y_j|}{2} = |x_j + y_j|,$$

which is a contradiction again. So $V \cap (-W) = V \cap f_{-1}(W) = \emptyset$ as well.

Overall, $\mathbb{Z}_2$ acts freely and properly discontinuously by diffeomorphisms on $\mathbb{R}^n \setminus \{0\}$. Hence it acts freely and properly discontinuously by diffeomorphisms on any manifold $M \subseteq \mathbb{R}^n \setminus \{0\}$ with $-M = M$.

(a) $0 \notin \mathcal{S}^n$ and $-\mathcal{S}^n = \mathcal{S}^n$, so $\mathcal{S}^n/\mathbb{Z}_2$ is an n-dimensional manifold, which is (diffeomorphic to) $\mathbb{RP}^n$.

(b) 0 is not in the cylinder $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, -1 < x_3 < 1\}$ and $-C = C$. Hence $C/\mathbb{Z}_2$ is a 2-dimensional manifold called the *Möbius band.*

(c) Similarly, $\mathbb{Z}_2$ acts freely and properly discontinuously on the torus T^2 in $\mathbb{R}^3$ and hence $T^2/\mathbb{Z}_2$ is a 2-dimensional manifold K called the *Klein bottle.*

Example. If we define for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ a map $f_a : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_a(x_1, \ldots, x_n) = (x_1 + a_1, \ldots, x_n + a_n),$$

this gives a homomorphism $\mathbb{Z}^n \to \mathrm{Diff}(\mathbb{R}^n)$ by $a \mapsto f_a$, which is a free and properly discontinuous group action. We therefore have that $\mathbb{R}^n/\mathbb{Z}^n$ is an n-dimensional manifold, which is diffeomorphic to T^n.

The quotient construction gives us an automatic example of a local diffeomorphism which is usually not a diffeomorphism.

Proposition 1.7. *If a discrete group G acts freely and properly discontinuously on M then the projection $\pi : M \to M/G$ is a surjective local diffeomorphism.*

Proof. **(Not examinable).** The projection map is clearly surjective. Let $p \in M$ and use the notation in the proof of Theorem 1.6. Then $p \in V_i$ for some i and hence $\pi(p) \in U_i$. So (V_i, ψ_i) is a chart around p and (U_i, φ_i) is a chart around $\pi(p)$. Then $\pi|_{V_i} = \pi_i$ so on $\psi_i(V_i)$ we have:

$$\varphi_i \circ \pi \circ \psi_i^{-1} = \varphi_i \circ \pi_i \circ \psi_i^{-1} = \psi_i \circ \pi_i^{-1} \circ \pi_i \circ \psi_i^{-1} = \mathrm{id},$$

which is a diffeomorphism. Since φ_i and ψ_i are diffeomorphisms onto their images we conclude that π_i is a diffeomorphism. Hence π is a local diffeomorphism at p. ☐

Remark. (Not examinable). Actually, π is a bit more than a surjective local diffeomorphism: it is a smooth covering map.

Example. We have local diffeomorphisms from $\mathcal{S}^n$ to $\mathbb{RP}^n$, the cylinder to the Möbius band and from T^2 to the Klein bottle, but they are *not* diffeomorphisms (although $\mathcal{S}^1$ is diffeomorphic to $\mathbb{RP}^1$).

Example. There is a version of the quotient construction for Lie groups acting on manifolds, which is a bit more complicated. However, it applies to the case of the group $\mathrm{U}(1)$ acting on the unit sphere $\mathcal{S}^{2n-1} \subseteq \mathbb{C}^{n+1}$ by multiplication. Recall that $\mathrm{U}(1) = \{e^{i\theta} \in \mathbb{C} : \theta \in \mathbb{R}\}$ and so we can act on $\mathbb{C}^{n+1}$ by $e^{i\theta} : z = (z_1, \ldots, z_{n+1}) \mapsto (e^{i\theta}z_1, \ldots, e^{i\theta}z_{n+1})$. We see that the quotient space $\mathcal{S}^{2n-1}/\,\mathrm{U}(1)$ is exactly $\mathbb{CP}^n$.

2 Tangent vectors

Let M be an n-dimensional manifold. We want to understand what we mean by tangent vectors to M. Again the definition is rather abstract, so let us postpone that and focus on some examples where we can compute the tangent space.

2.1 Tangent vectors and regular values

When we are working with manifolds which lie in some Euclidean space it is straightforward to make sense of tangent vectors. For example, for a curve in the plane $\alpha : \mathbb{R} \to \mathbb{R}^2$ (or into $\mathbb{R}^n$), it is just the line tangent to the curve, which we can calculate by writing

$$\alpha(t) = (a_1(t), a_2(t))$$

and computing the derivative

$$\alpha'(t) = (a_1'(t), a_2'(t)),$$

so the tangent vector at $\alpha(0) = p$, say, is

$$\alpha'(0) = (a_1'(0), a_2'(0)).$$

For a surface M in $\mathbb{R}^3$ a tangent vector at p is just a vector in $\mathbb{R}^3$ which is tangent to the surface. What does this really mean? Well, we could take a curve in the surface $\alpha : \mathbb{R} \to M \subseteq \mathbb{R}^3$ with $\alpha(0) = p$ and write

$$\alpha(t) = (a_1(t), a_2(t), a_3(t)) \in \mathbb{R}^3,$$

so then the tangent vector to the curve at p is

$$\alpha'(0) = (a_1'(0), a_2'(0), a_3'(0)).$$

By varying over all possible curves in M through p we get all of the tangent vectors to the curve at p. These tangent vectors will span a 2-dimensional plane: the plane tangent to M at p.

We can do the same trick if M is an n-dimensional manifold in $\mathbb{R}^{n+m}$. If we look at curves in M through p then the tangent vectors will form a vector space of dimension n which we denote by T_pM: the tangent space to M at p. In particular, we can calculate the tangent space to M at p for manifolds given by the regular value theorem.

Proposition 2.1. *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth map and let c be a regular value of F, so that $M = F^{-1}(c)$ is an n-dimensional manifold. Then for all $p \in M$, $T_pM \cong \operatorname{Ker} \mathrm{d}F_p$.*

Proof. Let $p \in M = F^{-1}(c)$ and let α be a curve in M through p. Then

$$F(\alpha(t)) = c \quad \text{for all } t$$

since $\alpha(t) \in F^{-1}(c)$ for all t. Differentiating both sides we see that

$$\frac{d}{dt}F(\alpha(t)) = 0.$$

Applying the Chain rule at $t = 0$, we see that

$$\mathrm{d}F_{\alpha(0)}(\alpha'(0)) = \mathrm{d}F_p(\alpha'(0)) = 0.$$

Hence $\alpha'(0) \in \operatorname{Ker} \mathrm{d}F_p$.

We thus have a linear map $T_pM \to \operatorname{Ker} \mathrm{d}F_p$. This map is clearly injective. Since c is a regular value, we know by the rank-nullity theorem that $\dim \operatorname{Ker} \mathrm{d}F_p = n + m - m = n$, so since T_pM is also n-dimensional the map must be surjective. □

We can now apply this result in a series of examples.

Example. We can write $\mathcal{S}^n = F^{-1}(0)$ where $F(x) = \sum_{i=1}^{n+1} x_i^2 - 1$. We saw that

$$\mathrm{d}F_x = (2x_1 \ \dots \ 2x_{n+1})$$

so

$$\operatorname{Ker} \mathrm{d}F_x = \{y \in \mathbb{R}^{n+1} \,:\, \langle y, x\rangle = 0\} = \langle x\rangle^{\perp},$$

the orthogonal complement of the line through x. Thus $T_x\mathcal{S}^n \cong \langle x\rangle^{\perp}$, which is geometrically clear.

Example. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be given by $F(x, y) = f(x)-y$. It is straightforward to calculate that $\mathrm{d}F_{(x,y)} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is given by

$$\mathrm{d}F_{(x,y)} = (\mathrm{d}f_x \ \ -I),$$

which clearly has rank m and thus is surjective. Hence

$$F^{-1}(0) = \operatorname{Graph}(f) = \{(x, f(x)) \,:\, x \in \mathbb{R}^n\}$$

is an n-dimensional manifold. We also see that

$$\operatorname{Ker} \mathrm{d}F_{(x,y)} = \{(u, v) \in \mathbb{R}^{n+m} \,:\, \mathrm{d}f_x(u) = v\} = \operatorname{Graph}(\mathrm{d}f_x)$$

and thus $T_{(x,f(x))}\operatorname{Graph}(f) \cong \operatorname{Graph}(\mathrm{d}f_x) \subseteq \mathbb{R}^{n+m}$.

In particular, for the paraboloid $M = \operatorname{Graph}(x_1^2 + x_2^2) \subseteq \mathbb{R}^3$, $T_pM \cong \operatorname{Span}\{(1, 0, 2x_1), (0, 1, 2x_2)\}$ for $p = (x_1, x_2, x_1^2 + x_2^2)$.

We again have a more sophisticated example.

Example. The set $\mathrm{SL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \,:\, \det(A) = 1\}$ is $F^{-1}(0)$ where $F : M_n(\mathbb{R}) \to \mathbb{R}$ is $F(A) = \det(A) - 1$ (which is smooth). Now, if A is invertible,

$$F(A + B) - F(A) = \det(A + B) - \det(A) = \det(A(I + A^{-1}B)) - \det(A) = \det(A)(\det(I + A^{-1}B) - 1)$$

and by expanding one sees that

$$\det(I + A^{-1}B) = 1 + \operatorname{tr}(A^{-1}B) + O(|B|^2).$$

Thus

$$\mathrm{d}F_A(B) = \operatorname{tr}(A^{-1}B)$$

for $A \in \mathrm{SL}_n(\mathbb{R})$, which is not the zero map since $\mathrm{d}F_A(A) = n$, for example. Hence Theorem 1.1 implies that $\mathrm{SL}(n, \mathbb{R})$ is an $(n^2 - 1)$-dimensional manifold. Moreover,

$$T_A \mathrm{SL}(n, \mathbb{R}) = \{B \in M_n(\mathbb{R}) \,:\, \operatorname{tr}(A^{-1}B) = 0\} \quad \Rightarrow \quad T_I \mathrm{SL}(n, \mathbb{R}) = \{B \in M_n(\mathbb{R}) \,:\, \operatorname{tr}(B) = 0\}.$$

This says that the *Lie algebra* $\mathfrak{sl}(n, \mathbb{R}) = T_I \mathrm{SL}(n, \mathbb{R})$ of the Lie group $\mathrm{SL}(n, \mathbb{R})$ (as a vector space) is the trace-free matrices. In fact the Lie bracket operation on the Lie algebra is just the matrix commutator, which is true of all matrix Lie groups with matrix Lie algebras.

We can also use tangent spaces to see that the cone we saw before is not a manifold.

Example. (Not examinable). If C is the cone $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_3^2 = x_1^2 + x_2^2\}$, then we have curves $(t, 0, t)$ and $(0, t, t)$ in C through 0 so $(1, 0, 1)$ and $(0, 1, 1)$ are tangent vectors to C at 0. However,

$$(1, -1, 0) = (1, 0, 1) - (0, 1, 1) \in \operatorname{Span}\{(1, 0, 1), (0, 1, 1)\}$$

but is not tangent to C at 0. This shows that C cannot be a 2-dimensional manifold because its "tangent space at 0" would have to be $\operatorname{Span}\{(1, 0, 1), (0, 1, 1)\}$, but we see that the tangent vectors to curves through 0 in C do not form a vector space.

2.2 Tangent vectors as differential operators

We now have an idea of what some tangent spaces look like, but we used the ambient space to do it. Therefore, for abstract manifolds it is not completely clear what we mean by a tangent vector or tangent space and it turns out to be a bit confusing on first sight.

Tangent vectors are some of the most important things to understand about manifolds, so we shall think hard about the definition.

The idea is to think about functions. So suppose we have a curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ through $p \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Then $f \circ \alpha : \mathbb{R} \to \mathbb{R}$ is a smooth function and we can differentiate it at 0:

$$(f \circ \alpha)'(0) = a_1'(0)\frac{\partial f}{\partial x_1}(p) + a_2'(0)\frac{\partial f}{\partial x_2}(p)$$

by the Chain rule. We therefore have a map $f \mapsto (f \circ \alpha)'(0)$ from functions to $\mathbb{R}$ given by

$$f \mapsto (a_1'(0)\frac{\partial}{\partial x_1}|_p + a_2'(0)\frac{\partial}{\partial x_2}|_p)f,$$

which is a differential operator acting on functions. If we think of $\{\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial x_2}|_p\}$ as a basis for a 2-dimensional vector space, then we identify this map with $\alpha'(0) = (a_1'(0), a_2'(0))$, which is the tangent vector to α at p that we saw before.

The good thing about this is that we can replace $\mathbb{R}^2$ by any manifold M, since $f \circ \alpha : \mathbb{R} \to \mathbb{R}$ can be differentiated, so this definition still works. Explicitly, if $\alpha : \mathbb{R} \to M$ is a curve through $p \in M$ and $f : M \to \mathbb{R}$ is a smooth function then we let (U, φ) be a coordinate chart at p and write $\varphi \circ \alpha(t) = (a_1(t), \ldots, a_n(t)) \in \varphi(U) \subseteq \mathbb{R}^n$. Then

$$\begin{aligned}(f \circ \alpha)'(0) &= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \alpha)(t)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi^{-1} \circ \varphi \circ \alpha)(t)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi^{-1})(a_1(t), \ldots, a_n(t))|_{t=0} \\ &= \sum_{j=1}^n a_j'(0)\frac{\partial (f \circ \varphi^{-1})}{\partial x_j}(\varphi(p)) = \left(\sum_{j=1}^n a_j'(0)\frac{\partial}{\partial x_j}|_{\varphi(p)}\right)(f \circ \varphi^{-1}).\end{aligned}$$

Hence, using the $\frac{\partial}{\partial x_j}|_{\varphi(p)}$ as a basis, we can identify the tangent vector to the curve $\varphi \circ \alpha$ in $\mathbb{R}^n$ at $\varphi(p)$ with the differential operator $\sum_{j=1}^n a_j'(0)\frac{\partial}{\partial x_j}|_{\varphi(p)}$ acting on the function $f \circ \varphi^{-1}$ (which is how we identify functions on M locally with functions on $\mathbb{R}^n$).

Notice that $\frac{\partial}{\partial x_j}|_{\varphi(p)}$ is the tangent vector to $t \mapsto \varphi^{-1}(0, \ldots, 0, t, 0, \ldots, 0)$, which is the image of a straight line, and forms a local basis for the tangent vectors to curves by the above calculation.

We have thus motivated the definition of tangent vector.

Definition 2.2. Let α be a curve in M through p, let $U \ni p$ be open in M and let $f : U \subseteq M \to \mathbb{R}$ be smooth at p. Then $f \circ \alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \to \mathbb{R}$ is smooth at 0 and we call

$$\alpha'(0) : f \mapsto (f \circ \alpha)'(0) \in \mathbb{R}$$

the *tangent vector* to α at 0. In other words, $\alpha'(0)$ is an operator which sends smooth functions f to the real number $(f \circ \alpha)'(0)$.

We say that X is a *tangent vector* to M at p if there exists a curve α in M through p such that the tangent vector to α at 0 is $\alpha'(0) = X$.

Remark. In this definition, tangent vectors are essentially differential operators on locally defined functions on M. We can also think of it as a vector in $\mathbb{R}^n$, using the given chart (U, φ) as described above.

Definition 2.3. We let T_pM denote the set of tangent vectors to M at p and we call T_pM the *tangent space* to M at p.

Proposition 2.4. *The tangent space T_pM to M at p is an n-dimensional vector space.*

Proof. **(Not examinable)** This is immediate from the observation that given p and a chart (U, φ) we can identify any tangent vector with a linear combination of $\frac{\partial}{\partial x_j}|_{\varphi(p)}$. ☐

2.3 Differential

Tangent spaces are extremely useful. In particular, they allow us to understand how to differentiate maps between manifolds as follows.

Definition 2.5. Let $f : M \to N$ be a smooth map between manifolds. Let $X = \alpha'(0) \in T_pM$. Then $f \circ \alpha$ is a curve in N through $f(p)$. We define the *differential* of f at p, which is a linear map $\mathrm{d}f_p : T_pM \to T_{f(p)}N$, by $\mathrm{d}f_p(X) = (f \circ \alpha)'(0)$.

We need to check this makes sense. So suppose $X = \alpha'(0) = \beta'(0)$, and we want to make sure $(f \circ \alpha)'(0) = (f \circ \beta)'(0)$. This is the same as saying that, for any smooth function h defined near $f(p)$ on N, $(h \circ f \circ \alpha)'(0) = (h \circ f \circ \beta)'(0)$. But $h \circ f$ is a smooth map defined near p on M, so $\alpha'(0) : h \circ f \mapsto (h \circ f \circ \alpha)'(0)$ and $\beta'(0) : h \circ f \mapsto (h \circ f \circ \beta)'(0)$. We assumed that $X = \alpha'(0) = \beta'(0)$, so it is well-defined.

We can also think of the differential in terms of a differential of a map between Euclidean spaces. Given a curve α through p and a chart (U, φ) at p, we have the curve $a = \varphi \circ \alpha$ in Euclidean space. The curve $f \circ \alpha$ defines a curve $b = \psi \circ f \circ \alpha$ in Euclidean space where (V, ψ) is a chart at $f(p)$. The relationship between the tangent vectors between the curves a and b at 0 is:

$$b'(0) = (\psi \circ f \circ \alpha)'(0) = (\psi \circ f \circ \varphi^{-1} \circ a)'(0) = \mathrm{d}(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}(a'(0)).$$

Hence the differential $\mathrm{d}f_p$ may be viewed as $\mathrm{d}(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}$ given the charts.

In particular, this means that if $M \subseteq \mathbb{R}^n$ and $N \subseteq \mathbb{R}^m$ are manifolds and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map such that $f(M) \subseteq N$ then $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is the restriction of the linear map $\mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}^m$.

Example. Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ be given by $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Then we see that

$$\mathrm{d}f_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

If we let $\partial_r, \partial_\theta$ denote differentiation with respect to r, θ and ∂_1, ∂_2 be differentiation with respect to x_1, x_2 on $\mathbb{R}^2$ then

$$\mathrm{d}f_{(r,\theta)}(\partial_r) = \cos\theta\partial_1 + \sin\theta\partial_2$$

and

$$\mathrm{d}f_{(r,\theta)}(\partial_\theta) = -r\sin\theta\partial_1 + r\cos\theta\partial_2.$$

This is nothing but a restatement of the Chain rule for differentiating f with respect to r, θ, and the differential is the Jacobian for the transformation to polar coordinates.

Example. Let $f : \mathbb{R}^2 \to \mathcal{S}^2 \subseteq \mathbb{R}^3$ be given by $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, then

$$\mathrm{d}f_{(\theta,\phi)} = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

In terms of differential operators we therefore have that

$$\mathrm{d}f_{(\theta,\phi)}(\partial_\theta) = \cos\theta\cos\phi\partial_1 + \cos\theta\sin\phi\partial_2 - \sin\theta\partial_3 \quad \text{and} \quad \mathrm{d}f_{(\theta,\phi)}(\partial_\phi) = -\sin\theta\sin\phi\partial_1 + \sin\theta\cos\phi\partial_2.$$

Example. Let $f : \mathbb{R}^n \to T^n$ be given by $f(\theta_1, \ldots, \theta_n) = (\cos\theta_1, \sin\theta_1, \ldots, \cos\theta_n, \sin\theta_n)$. Then

$$\mathrm{d}f_{(\theta_1,\ldots,\theta_n)}\partial_{\theta_j} = -\sin\theta_j\partial_{2j-1} + \cos\theta_j\partial_{2j}.$$

Example. Let us calculate the differential of the map $f : \mathcal{S}^2 \to \mathbb{RP}^2$ given by $f(x) = [x]$ at $(0,0,1) \in U_S$. Let $X \in T_{(0,0,1)}\mathcal{S}^2$. Then $f(0,0,1) = [(0,0,1)] \in U_3 = \{[(y_1,y_2,y_3)] \in \mathbb{RP}^2 : y_3 \neq 0\}$, so we want to calculate $\mathrm{d}f_{(0,0,1)}(X)$. Now, we know that $\varphi_S(0,0,1) = (0,0)$ and for $x = (x_1,x_2) \in \mathbb{R}^2$ with $|x| < 1$,

$$\varphi_3 \circ f \circ \varphi_S^{-1}(x_1,x_2) = \varphi_3\left[\left(\frac{2x_1}{1+|x|^2}, \frac{2x_2}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right)\right] = \left(\frac{2x_1}{1-|x|^2}, \frac{2x_2}{1-|x|^2}\right).$$

Hence

$$\mathrm{d}(\varphi_3 \circ f \circ \varphi_S^{-1})|_{(0,0)} = \frac{2}{(1-|x|^2)^2}\begin{pmatrix} 1+x_1^2-x_2^2 & 2x_1x_2 \\ 2x_1x_2 & 1-x_1^2+x_2^2 \end{pmatrix}|_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Therefore we can view $\mathrm{d}f_{(0,0,1)}$ as $2\,\mathrm{id}$ using these charts.

Since the differential can be identified with a differential in Euclidean space, it means that we have the following basic properties of the differential because they hold in Euclidean space.

Proposition 2.6. *(a) The identity map* $\mathrm{id} : M \to M$ *satisfies* $\mathrm{d}\,\mathrm{id}_p = \mathrm{id} : T_pM \to T_pM$ *for all* $p \in M$.

(b) If $f : P \to N$ *and* $g : M \to P$ *are smooth maps then* $f \circ g : M \to N$ *satisfies the Chain rule:*

$$\mathrm{d}(f \circ g)_p = \mathrm{d}f_{g(p)} \circ \mathrm{d}g_p.$$

Example. Let $f : M \to N$ be a diffeomorphism. Then $f^{-1} \circ f = \mathrm{id}$ and $f \circ f^{-1} = \mathrm{id}$. Hence by the Chain rule and Proposition 2.6 we see that

$$\mathrm{d}(f^{-1})_{f(p)} \circ \mathrm{d}f_p = \mathrm{id} = \mathrm{d}f_p \circ \mathrm{d}(f^{-1})_{f(p)}$$

for all $p \in M$, so $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is invertible with inverse

$$(\mathrm{d}f_p)^{-1} = \mathrm{d}(f^{-1})_{f(p)}$$

2.4 Local diffeomorphisms

We can use the differential of f at p to detect when f is a local diffeomorphism. This is a very important result because knowing when a given map f is a local diffeomorphism is difficult on inspection as it is nonlinear in general, but the differential is a linear map and so is easier to analyse.

Proposition 2.7. *A smooth map* $f : M \to N$ *is a local diffeomorphism at* p *if and only if* $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ *is an isomorphism.*

Proof. **(Not examinable).** Suppose that f is a local diffeomorphism at p. Then there exist open $U \ni p$ and open $V \ni f(p)$ such that $f : U \to V$ is a diffeomorphism.

Thus $\mathrm{d}(f^{-1} \circ f)_p = \mathrm{d}(f^{-1})_{f(p)} \circ \mathrm{d}f_p = \mathrm{id}$ and $\mathrm{d}f_p \circ \mathrm{d}(f^{-1})_{f(p)} = \mathrm{id}$. Hence $\mathrm{d}f_p$ is an isomorphism.

Now suppose that $\mathrm{d}f_p$ is an isomorphism. Let (U,φ) and (V,ψ) be charts around p and $f(p)$ respectively so that $f(U) \subseteq V$.

Then by the first part of the proof $\mathrm{d}\varphi^{-1}_{\varphi(p)} : \mathbb{R}^n \to T_pM$ and $\mathrm{d}\psi_{f(p)} : T_{f(p)}N \to \mathbb{R}^n$ are isomorphisms since φ^{-1} and ψ are local diffeomorphisms, so $\mathrm{d}(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} : \mathbb{R}^n \to \mathbb{R}^n$ is a composition of isomorphisms by the chain rule and thus is an isomorphism: explicitly,

$$\mathrm{d}(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = \mathrm{d}\psi_{f(p)} \circ \mathrm{d}f_p \circ \mathrm{d}(\varphi^{-1})_{\varphi(p)}.$$

We can then use the Inverse Function Theorem to give open sets $U' \ni p$ and $V' \ni f(p)$ so that $\psi \circ f \circ \varphi^{-1} : \varphi(U') \to \psi(V')$ is a diffeomorphism (using the fact that φ, ψ are diffeomorphisms onto their images). Hence $f : U' \to V'$ is a diffeomorphism. $\square$

Remark. As can be seen from the proof, Proposition 2.7 is essentially the manifold version of the Inverse Function Theorem.

Example. The map $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ satisfied

$$\mathrm{d}f_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

which always has non-zero determinant $r > 0$. Therefore f is a local diffeomorphism (which is not a diffeomorphism because it is not injective).

Example. The map $f : \mathbb{R}^2 \to \mathcal{S}^2$ satisfied

$$\mathrm{d}f_{(\theta,\phi)} = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

which has full rank (and is therefore an isomorphism) except when $\sin\theta = 0$. Hence f is not a local diffeomorphism, but it is one restricted to any region where $\sin\theta \neq 0$, so $\theta \in (0, \pi)$ for example.

Example. The map $f : \mathbb{R}^n \to T^n$ clearly has differential whose image is n-dimensional and thus is an isomorphism, and hence f is a local diffeomorphism.

Example. The map $f : \mathcal{S}^n \to \mathbb{RP}^n$ given by $f(x) = [x]$ has the property that for any $p \in \mathcal{S}^n$, $\mathrm{d}f_p$ can be identified with $2\,\mathrm{id}$ using appropriate charts, so f is a local diffeomorphism. It is not a diffeomorphism because it is not a bijection.

Example. The map $f : \mathbb{R}^2 \to T^2 \subseteq \mathbb{R}^3$ given by $f(\theta,\phi) = ((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta)$ satisfies

$$\mathrm{d}f_{(\theta,\phi)} = \begin{pmatrix} -\sin\theta\cos\phi & -(2+\cos\theta)\sin\phi \\ -\sin\theta\sin\phi & (2+\cos\theta)\cos\phi \\ \cos\theta & 0 \end{pmatrix},$$

which is always full rank, so $\mathrm{d}f_{(\theta,\phi)}$ is always an isomorphism. Hence f is a local diffeomorphism. However it is not a diffeomorphism because it is clearly not a bijection.

2.5 Regular values

We now have another useful application of the differential, which is the manifold version of the regular value theorem. The proof is immediate by adapting the proof of the usual regular value theorem.

Theorem 2.8. (Regular value theorem). *Let M be a manifold of dimension $m+n$ and let N be a manifold of dimension m. Suppose that $F : M \to N$ is smooth and let $c \in N$ be such that $F^{-1}(c) \neq \emptyset$ and $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is surjective for all $p \in F^{-1}(c)$ (i.e. c is a regular value for F). Then $F^{-1}(c)$ is an n-dimensional manifold and $T_pF^{-1}(c) = \mathrm{Ker}\,\mathrm{d}F_p$ for all $p \in F^{-1}(c)$.*

Remark. (Not examinable). Sard's theorem implies that, given a smooth map $F : M \to N$ between manifolds, if $\mathcal{C}$ is the set of points p in M where $\mathrm{d}F_p$ is not surjective, then $F(\mathcal{C})$ (i.e. the critical values of F) has measure zero in N. This means that almost every point in the image of F is a regular value.

Example. Let $F : \mathcal{S}^n \to \mathbb{R}$ be given by $F(x_1, \ldots, x_{n+1}) = x_{n+1}$. Then, as a map from $\mathbb{R}^{n+1} \to \mathbb{R}$, we have

$$\mathrm{d}F_x = (0 \ \ldots \ 0 \ 1),$$

which is non-zero on $T_x\mathcal{S}^n$ except at points where $x = (0, \ldots, 0, \pm 1)$, i.e. when $F(x) = \pm 1$. Hence $F^{-1}(c)$ is an $n-1$-dimensional manifold for all $c \in (-1, 1)$ (all diffeomorphic to $\mathcal{S}^{n-1}$).

We also see that $T_xF^{-1}(c) = \{y \in T_x\mathcal{S}^n : y_{n+1} = 0\}$ (i.e. tangent vectors to $\mathcal{S}^n$ at x which are orthogonal to the "vertical" x_{n+1} direction).

Example. Suppose we know that $\mathrm{U}(n)$ is an n^2-dimensional manifold and that we want to show that $\mathrm{SU}(n)$ is n^2-1-dimensional manifold. The map $F : \mathrm{U}(n) \to \mathcal{S}^1 \subseteq \mathbb{C}$ given by $F(A) = \det A$ satisfies

$$\mathrm{d}F_A(B) = \det A \operatorname{tr}(A^{-1}B).$$

If we consider $A \in \mathrm{SU}(n)$, so $\det A = 1$, we want to show $\operatorname{tr}(A^{-1}B)$ is non-zero where

$$B \in T_A \mathrm{U}(n) = \{B \in M_n(\mathbb{C}) \,:\, \overline{A^{\mathrm{T}}}B + \overline{B^{\mathrm{T}}}A = 0\}.$$

Take $B = iA$, then $B \in T_A \mathrm{U}(n)$ and $\mathrm{d}F_A(B) = \operatorname{tr}(iI) = ni \neq 0$. Thus, $\mathrm{SU}(n)$ is an n^2-1-dimensional manifold and

$$T_I \mathrm{SU}(n) = \{B \in M_n(\mathbb{C}) \,:\, \overline{B^{\mathrm{T}}} = -B,\, \operatorname{tr}(B) = 0\}.$$

This describes the Lie algebra $\mathfrak{su}(n) = T_I \mathrm{SU}(n)$ of the Lie group $\mathrm{SU}(n)$.

2.6 Immersions, embeddings and submersions

We can also use the differential to define special types of maps.

Definition 2.9. A smooth map $f : M \to N$ is an immersion if $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is injective for all $p \in M$ (so we obviously need $\dim N \geq \dim M$).

An injective immersion which is a homeomorphism onto its image is called an embedding. If M is compact, then an injective immersion is an embedding. If $f : M \to N$ is an embedding then $f(M)$ is a manifold and $f : M \to f(M)$ is a diffeomorphism. In this case, we say that $f(M)$ (or M) is a submanifold of N. Many of the examples of manifolds we have seen are submanifolds of some Euclidean space, where f was the inclusion map.

A smooth map $f : M \to N$ is a submersion if $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is surjective for all $p \in M$ (so we obviously need $\dim N \leq \dim M$).

A map which is both an immersion and a submersion is a local diffeomorphism.

Example. The map $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(\theta) = (\cos\theta, \sin\theta)$ satisfies

$$\mathrm{d}f_\theta(\partial_\theta) = -\sin\theta\partial_1 + \cos\theta\partial_2$$

which is non-zero for all θ, so $\mathrm{d}f_\theta$ is injective for all θ. Hence f is an immersion.

The map f is not an embedding since $f(\theta + 2\pi) = f(\theta)$.

Define a free and properly discontinuous group action of $\mathbb{Z}$ on $\mathbb{R}$ by $f_n(\theta) = \theta + 2\pi n$ for $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$. Then the map $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ given by $F([\theta]) = f(\theta)$ is well-defined since $f \circ f_n(\theta) = f(\theta + 2\pi n) = f(\theta)$ for all $n \in \mathbb{Z}$. If $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the projection map then $f = F \circ \pi$ so

$$\mathrm{d}f_\theta = \mathrm{d}F_{[\theta]} \circ \mathrm{d}\pi_\theta.$$

Since π is a local diffeomorphism, $\mathrm{d}\pi_\theta$ is an isomorphism, so $\mathrm{d}f_\theta$ is injective if and only if $\mathrm{d}F_{[\theta]}$ is injective.

We deduce that $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ is an immersion which is now injective, so F is an embedding. We then have that $F(\mathbb{R}/\mathbb{Z}) = \mathcal{S}^1$ and $\mathcal{S}^1 \cong \mathbb{R}/\mathbb{Z}$ (i.e. they are diffeomorphic).

Example. Let $C = \{(\cos\theta, \sin\theta, t) \in \mathbb{R}^3 \,:\, t \in \mathbb{R}\}$ be the cylinder. Let $f : \mathcal{S}^1 \to C$ be given by $f(\cos\theta, \sin\theta) = (\cos\theta, \sin\theta, 0)$. Then

$$T_{(\cos\theta,\sin\theta)}\mathcal{S}^1 = \{\lambda(-\sin\theta\partial_1 + \cos\theta\partial_2) \,:\, \lambda \in \mathbb{R}\}$$

and

$$\mathrm{d}f_{(\cos\theta,\sin\theta)}(-\sin\theta\partial_1 + \cos\theta\partial_2) = -\sin\theta\partial_1 + \cos\theta\partial_2,$$

so f is an immersion. Moreover, f is injective so f is an embedding.

Now let $g : C \to \mathcal{S}^1$ be given by $g(\cos\theta, \sin\theta, t) = (\cos\theta, \sin\theta)$. Then

$$T_{(\cos\theta,\sin\theta,t)}C = \mathrm{Span}\{-\sin\theta\partial_1 + \cos\theta\partial_2, \partial_3\}$$

and

$$\begin{aligned} \mathrm{d}g_{(\cos\theta,\sin\theta,t)}(-\sin\theta\partial_1 + \cos\theta\partial_2) &= -\sin\theta\partial_1 + \cos\theta\partial_2, \\ \mathrm{d}g_{(\cos\theta,\sin\theta,t)}(\partial_3) &= 0. \end{aligned}$$

Hence g is a submersion.

Example. The map $F : \mathcal{S}^n \to \mathbb{R}$ given by $F(x_1, \ldots, x_{n+1}) = x_{n+1}$ is not a submersion because $\mathrm{d}F_p$ is the zero map at the North and South poles. However, $F : \mathcal{S}^n \setminus \{N, S\} \to \mathbb{R}$ is a submersion.

This indicates the relationship between submersions and regular values.

Example. The map $F : \mathrm{U}(n) \to \mathcal{S}^1 \subseteq \mathbb{C}$ given by $F(A) = \det A$ is a submersion. We already showed it is a submersion at all points where $\det A = 1$, but it is also true at all other points in $\mathrm{U}(n)$.

Example. The map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ given by $\pi(z) = [z]$ is a submersion.

The map $\pi : \mathcal{S}^{2n+1} \subseteq \mathbb{C}^{n+1} \to \mathbb{CP}^n$ given by $\pi(z) = [z]$ is also a submersion, called the Hopf fibration.

3 Vector fields

We now want to build "global" objects called vector fields out of tangent vectors.
Fake definition: A vector field is a choice of tangent vector at each point, which varies smoothly.

Again, although this definition is fake, it gives the right idea. Hence, we have that vector fields are differential operators on smooth functions. On $\mathbb{R}^n$, a vector field is always of the form

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} a_i \partial_i,$$

where $a_i : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. This is always locally what vector fields look like.

3.1 Tangent bundle

To make the definition more precise, we see that the tangent spaces give a family of $\mathbb{R}^n$'s which lie "over" the manifold M. We can build a manifold out of the tangent spaces, which shows that the tangent spaces fit together in a natural way, and from this structure it will then be clear what a vector field is.

Definition 3.1. We define the *tangent bundle* TM of M to be

$$TM = \cup_{p\in M} T_pM.$$

Theorem 3.2. *The tangent bundle TM is a $2n$-dimensional manifold such that*

- *there exists a smooth surjective map $\pi : TM \to M$ such that*
- *$\pi^{-1}(p) = T_pM$, which is a vector space for all $p \in M$, and*
- *for all $p \in M$ there exists an open set $U \ni p$ and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that $\psi : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^n$ is an isomorphism for all $q \in U$.*

Proof. **(Not examinable).** Let $\{(U_i, \varphi_i) : i \in I\}$ be an atlas for M and let $\pi : TM \to M$ be the natural projection $\pi(p, X) = p$.

- Let $V_i = \pi^{-1}(U_i)$ which we define to be open and clearly $\cup_{i\in I} V_i = TM$.
- Let $\psi_i : V_i \to \mathbb{R}^n \times \mathbb{R}^n$ be given by

  $$\psi_i(p, X) = \big(\varphi_i(p), \mathrm{d}(\varphi_i)_p(X)\big),$$

  so that $\psi_i : V_i \to U_i \times \mathbb{R}^n$ is a homeomorphism. It is clearly a bijection and continuous with continuous inverse because the same is true of φ_i and $\mathrm{d}(\varphi_i)_p$ is an isomorphism by Proposition 2.7.
- If $V_i \cap V_j \neq \emptyset$ then

  $$\psi_j \circ \psi_i^{-1}(q, u) = \big(\varphi_j \circ \varphi_i^{-1}(q), \mathrm{d}(\varphi_j)_{\varphi_i^{-1}(q)} \circ \mathrm{d}(\varphi_i)_q^{-1}(u)\big) = \big(\varphi_j \circ \varphi_i^{-1}(q), \mathrm{d}(\varphi_j \circ \varphi_i^{-1})_q(u)\big)$$

  The first factor is a diffeomorphism and the second factor is an isomorphism, so overall the transition map is a diffeomorphism.

We have satisfied the conditions of Definition 1.2 so TM is a $2n$-dimensional manifold.

The remaining conditions are clearly satisfied by construction. $\square$

Remark. (Not examinable). Theorem 3.2 is an example where it is more convenient to define the topology on TM using the atlas.

This gives yet another way to build manifolds out of given ones.

Example. Clearly $T_p\mathbb{R}^n = \mathbb{R}^n$ (canonically) for all $p \in \mathbb{R}^n$ and so $T\mathbb{R}^n = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

Example. It is straightforward to see that points in $T\mathcal{S}^1$ are given by $p = (\cos\theta, \sin\theta)$ and $q = \lambda(-\sin\theta, \cos\theta)$ since q must be orthogonal to p, for some $\lambda, \theta \in \mathbb{R}$. Hence, there is an obvious diffeomorphism $f : \mathcal{S}^1 \times \mathbb{R} \to T\mathcal{S}^1$ given by

$$f : (\theta, \lambda) \mapsto \lambda(-\sin\theta, \cos\theta) \in T_{(\cos\theta,\sin\theta)}\mathcal{S}^1.$$

Hence $T\mathcal{S}^1$ is diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}$.

Moreover, for fixed θ, $f : \lambda \to \lambda(-\sin\theta, \cos\theta) \in T_{(\cos\theta,\sin\theta)}\mathcal{S}^1$ is an isomorphism of vector spaces.

Example. $T\mathcal{S}^2 \neq \mathcal{S}^2 \times \mathbb{R}^2$: we shall see why later. We know that points in $T\mathcal{S}^2$ are given by $x \in \mathcal{S}^2$ and $y \in \mathbb{R}^3$ orthogonal to x. We have that x defines an oriented straight line through 0. Since y is orthogonal to x we can use it translate this straight line to get an oriented straight line through y in the direction x.

Conversely, given an oriented straight line in $\mathbb{R}^3$, there is a unique closest point from the line to 0, which gives a vector $y \in \mathbb{R}^3$ orthogonal to the line. Translating by y gives an oriented straight line through 0, which is uniquely determined by some $x \in \mathcal{S}^2$.

Hence the set of all oriented straight lines in $\mathbb{R}^3$ is a 4-dimensional manifold, which is $T\mathcal{S}^2$.

We can do the same in higher dimensions to describe $T\mathcal{S}^n$.

Example. Similarly, the set of all straight lines in $\mathbb{R}^3$ is a 4-dimensional manifold, which is $T\mathbb{RP}^2$.

3.2 Definition of vector fields

We now can make the real definition of vector fields.

Definition 3.3. A *vector field* X on a manifold M is a smooth map $X : M \to TM$ such that $X(p) \in T_pM$ for all $p \in M$. We denote the set of vector fields by $\Gamma(TM)$.

Example. On $\mathbb{R}^n$ we have standard vector fields $\partial_i = \frac{\partial}{\partial x_i}$. These are clearly differential operators, so if $f : \mathbb{R}^n \to \mathbb{R}$ then $\partial_i(f) : \mathbb{R}^n \to \mathbb{R}$ is $\frac{\partial f}{\partial x_i}$.

Remark. From now on we shall always use ∂_i to denote the standard vector fields on $\mathbb{R}^n$.

In general, given a smooth function $f : M \to \mathbb{R}$ and a vector field X on M, we will have that $X(f) : M \to \mathbb{R}$ is a smooth function, which we view as the derivative of f by X, just as in the example above.

Example. If $M \subseteq \mathbb{R}^n$ then a vector field X on M is a restriction of a vector field on $\mathbb{R}^n$ so that $X(p) \in T_pM$ for all $p \in M$.

For example, if we take $\mathcal{S}^1 \subseteq \mathbb{R}^2$ then if $(x_1, x_2) = (\cos\theta, \sin\theta) \in \mathcal{S}^1$ we have

$$T_{(x_1,x_2)}\mathcal{S}^1 = T_{(\cos\theta,\sin\theta)}\mathcal{S}^1 \cong \{\lambda(-\sin\theta, \cos\theta) : \lambda \in \mathbb{R}\} = \{(\lambda(-x_2, x_1) : \lambda \in \mathbb{R}\}.$$

Therefore, the vector field $X = -x_2\partial_1 + x_1\partial_2$ on $\mathbb{R}^2$ restricts to be a vector field on $\mathcal{S}^1$.

Example. We can define vector fields on $\mathbb{R}^3$ by

$$E_1 = x_3\partial_2 - x_2\partial_3,\ E_2 = x_1\partial_3 - x_3\partial_1,\ E_3 = x_2\partial_1 - x_1\partial_2.$$

Clearly, these vector fields should have something to do with circles in the $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ planes, based on the previous example. We will see this later.

Example. Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ be given by $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Then we saw that

$$\mathrm{d}f_{(r,\theta)}(\partial_r) = \cos\theta\partial_1 + \sin\theta\partial_2$$

and

$$\mathrm{d}f_{(r,\theta)}(\partial_\theta) = -r\sin\theta\partial_1 + r\cos\theta\partial_2.$$

We see that if we let

$$X_r = \cos\theta\partial_1 + \sin\theta\partial_2 = \frac{x_1\partial_1 + x_2\partial_2}{r}$$

and

$$X_\theta = -r\sin\theta\partial_1 + r\cos\theta\partial_2 = -x_2\partial_1 + x_1\partial_2,$$

then these are well-defined vector fields on $\mathbb{R}^2 \setminus \{0\}$. (Often we abuse notation and call $X_r = \partial_r$ and $X_\theta = \partial_\theta$.)

Example. Let $f : \mathbb{R}^2 \to \mathcal{S}^2 \subseteq \mathbb{R}^3$ be given by $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, then

$$\mathrm{d}f_{(\theta,\phi)}(\partial_\theta) = \cos\theta\cos\phi\partial_1 + \cos\theta\sin\phi\partial_2 - \sin\theta\partial_3 \quad \text{and} \quad \mathrm{d}f_{(\theta,\phi)}(\partial_\phi) = -\sin\theta\sin\phi\partial_1 + \sin\theta\cos\phi\partial_2.$$

We have that

$$X_\theta = \cos\theta\cos\phi\partial_1 + \cos\theta\sin\phi\partial_2 - \sin\theta\partial_3$$

and

$$X_\phi = -\sin\theta\sin\phi\partial_1 + \sin\theta\cos\phi\partial_2$$

are vector fields on $\mathcal{S}^2 \setminus \{N, S\}$. We can extend X_ϕ to $\mathcal{S}^2$ since it vanishes at N, S, but X_θ does not extend smoothly to N, S. Again, we usually say that $X_\theta = \partial_\theta$ and $X_\phi = \partial_\phi$.

Example. Let $f : \mathbb{R}^n \to T^n$ be given by $f(\theta_1, \ldots, \theta_n) = (\cos\theta_1, \sin\theta_1, \ldots, \cos\theta_n, \sin\theta_n)$. Then

$$\mathrm{d}f_{(\theta_1,\ldots,\theta_n)}\partial_{\theta_j} = -\sin\theta_j\partial_{2j-1} + \cos\theta_j\partial_{2j},$$

and we have that

$$X_j = -\sin\theta_j\partial_{2j-1} + \cos\theta_j\partial_{2j}$$

are vector fields on T^n for all j.

3.3 Parallelizable manifolds

Now, we can write all vector fields on $\mathbb{R}^n$ as linear combinations of the ∂_i, but when can we do this on a general manifold M? We saw above that it worked sometimes and sometimes not: for example, we did not seem able to do this globally on $\mathcal{S}^2$ unless the vector field vanished at the poles. This is a question of whether the tangent bundle is a product or not, so we make the following definition.

Definition 3.4. The tangent bundle TM of M is trivial if there exists a diffeomorphism $\psi : TM \to M \times \mathbb{R}^n$ such that $\psi : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ is an isomorphism for all $p \in M$; i.e. a bundle isomorphism between TM and $M \times \mathbb{R}^n$.

If TM is trivial we say that M is parallelizable.

Example. $\mathbb{R}^n$ is obviously parallelizable.

Example. $\mathcal{S}^1$ is parallelizable by our example earlier. We will see that $\mathcal{S}^3$ is parallelizable, but $\mathcal{S}^{2n}$ is not and $\mathcal{S}^5$ is not, for example.

Example. All of the matrix groups G we have seen are parallelizable. In fact, all Lie groups are parallelizable.

We can use vector fields to test whether a manifold is parallelizable. The proof is a standard exercise that you should certainly do, and we will see a more general version of this statement later.

Proposition 3.5. *An n-dimensional manifold is parallelizable if and only if it has n linearly independent vector fields (meaning that they are linearly independent at every point of M).*

Example. For a 1-dimensional manifold, being parallelizable is the same as having a nowhere vanishing vector field.

We see that the vector field $-\sin\theta\partial_1 + \cos\theta\partial_2$ on $\mathcal{S}^1$ is nowhere vanishing and hence we can confirm again that $\mathcal{S}^1$ is parallelizable.

Example. On $\mathcal{S}^n$ a vector field can be thought of as a map $X : \mathcal{S}^n \to \mathbb{R}^{n+1}$ such that $X(p) \in T_p\mathcal{S}^n = \langle p\rangle^\perp$ for all $p \in \mathcal{S}^n$.

To find linearly independent vector fields we certainly need to have that the vector fields are nowhere vanishing.

However, the Hairy Ball Theorem implies that every vector field on $\mathcal{S}^{2n}$ has at least one point where it vanishes, hence by Proposition 3.5 $T\mathcal{S}^{2n}$ is not trivial.

Example. T^n is parallelizable since the vector fields

$$X_j = -\sin\theta_j\partial_{2j-1} + \cos\theta_j\partial_{2j}$$

on T^n for $j = 1, \ldots, n$ are linearly independent.

3.4 Pushforward

Now, we said that locally vector fields always look like vector fields on $\mathbb{R}^n$. To make this precise, we show how to "push" vector fields from $\mathbb{R}^n$ into a manifold and, more generally, from one manifold to another.

Definition 3.6. Let $f : M \to N$ be a diffeomorphism. Then we define the *pushforward* $f_* : \Gamma(TM) \to \Gamma(TN)$ by

$$f_*(X)(f(p)) = \mathrm{d}f_p(X(p))$$

for all $p \in M$. This clearly defines a vector field $f_*(X)$ on N from a vector field X on M because f is a diffeomorphism.

Remark. (Not examinable). We can define the pushforward more generally but it does not really work for a general smooth map. First of all, if f is not injective the potential pushforward vector field is not even well-defined, and if f is not surjective then the vector field is not defined on all of N.

Example. We see in the case $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ given by $f(r,\theta) = (r\cos\theta, r\sin\theta)$ that $X_r = f_*(\partial_r)$ and $X_\theta = f_*(\partial_\theta)$.

Example. We see when $f : \mathbb{R}^2 \to \mathcal{S}^2 \subseteq \mathbb{R}^3$ is given by $f(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, then $X_\theta = f_*(\partial_\theta)$ and $X_\phi = f_*(\partial_\phi)$ where f is a diffeomorphism, so for example for $f : (0,\pi) \times (0,2\pi) \to \mathcal{S}^2$.

Example. Let $f : \mathbb{R}^n \to T^n$ be given by $f(\theta_1, \ldots, \theta_n) = (\cos\theta_1, \sin\theta_1, \ldots, \cos\theta_n, \sin\theta_n)$. Then $X_j = f_*(\partial_{\theta_j})$ for all j.

The local correspondence between vector fields on M and vector fields on $\mathbb{R}^n$ in a chart (U,φ) is nothing other than $X \mapsto \varphi_*(X)$ where we consider X restricted to U. Explicitly, if X is any vector field on M, then $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism, and

$$\varphi_*(X) = \sum_{i=1}^n a_i\partial_i$$

for some smooth functions $a_i : \varphi(U) \to \mathbb{R}$. This is very useful.

Perhaps even more useful is that $\varphi^{-1} : \varphi(U) \to U$ is a diffeomorphism, so

$$(\varphi^{-1})_*(\sum_{i=1}^n a_i\partial_i) = X$$

is a vector field on $U \subseteq M$, and every vector field on U can be given this way.

3.5 Lie bracket

We have seen that a vector field allows us to differentiate functions. Now, we would like to compose vector fields, just like we compose derivatives, but (as we will now see) there is a problem. Suppose X, Y are vector fields on $\mathbb{R}^n$ given by $\sum_{i=1}^n a_i\partial_i$ and $\sum_{i=1}^n b_i\partial_i$. Then the operator $X \circ Y$ is given by

$$\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}(\sum_{j=1}^n b_j \frac{\partial}{\partial x_j}) = \sum_{i,j=1}^n a_i b_j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n(\sum_{i=1}^n a_i \frac{\partial b_j}{\partial x_i})\frac{\partial}{\partial x_j}$$

which is not a linear combination of ∂_i and so is not a vector field on $\mathbb{R}^n$. However, it is clear, that if we look at

$$X \circ Y - Y \circ X = \sum_{j=1}^n(\sum_{i=1}^n a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i})\partial_j,$$

then this is a vector field on $\mathbb{R}^n$.

Definition 3.7. Given $X, Y \in \Gamma(TM)$ we define the *Lie bracket* of X, Y to be $[X, Y] = X \circ Y - Y \circ X$, i.e. if f is a smooth function on M then

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Then $[X, Y] \in \Gamma(TM)$.

Remark. Notice that $[Y, X] = -[X, Y]$ so $[X, X] = 0$.

By the calculation above we see that, if in a given chart (U, φ) we have

$$\varphi_*(X) = \sum_{i=1}^n a_i\partial_i \quad \text{and} \quad \varphi_*(Y) = \sum_{i=1}^n b_i\partial_i,$$

then

$$\varphi_*[X, Y] = [\varphi_*(X), \varphi_*(Y)] = \sum_{j=1}^n \left(\sum_{i=1}^n a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\partial_j.$$

Example. If ∂_i and ∂_j are standard vector fields on $\mathbb{R}^n$ then

$$[\partial_i, \partial_j] = \partial_i(\partial_j) - \partial_j(\partial_i) = \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_j \partial x_i} = 0$$

so $[\partial_i, \partial_j] = 0$.

Example. Let $E_1 = x_3\partial_2 - x_2\partial_3$, $E_2 = x_1\partial_3 - x_3\partial_1$ and $E_3 = x_2\partial_1 - x_1\partial_2$ be vector fields on $\mathbb{R}^3$. Then

$$[E_1, E_2] = (x_3\partial_2 - x_2\partial_3)(x_1\partial_3 - x_3\partial_1) - (x_1\partial_3 - x_3\partial_1)(x_3\partial_2 - x_2\partial_3) = (x_2\partial_1 - x_1\partial_2) = E_3$$

so $[E_1, E_2] = E_3$. Similarly $[E_2, E_3] = E_1$ and $[E_3, E_1] = E_2$.

Example. Let's take

$$X = x_1\partial_1 + x_2\partial_2 \quad \text{and} \quad Y = -x_2\partial_1 + x_1\partial_2.$$

Then

$$[X, Y] = (x_1\partial_1 + x_2\partial_2)(-x_2\partial_1 + x_1\partial_2) - (-x_2\partial_1 + x_1\partial_2)(x_1\partial_1 + x_2\partial_2) = x_1\partial_2 - x_2\partial_1 - (-x_2\partial_1 + x_1\partial_2) = 0.$$

Example. Clearly the vector fields $X_j = -\sin\theta_j\partial_{2j-1} + \cos\theta_j\partial_{2j}$ on T^n satisfy $[X_i, X_j] = 0$.

We have the following immediate facts about the Lie bracket, which makes calculation much easier.

Proposition 3.8. *Let $f : M \to N$ be a diffeomorphism. Then $f_*[X, Y] = [f_*X, f_*Y]$.*

Proof. **(Not examinable).** If we choose local charts (U, φ) and (V, ψ) for M and N such that $f : U \to V$ is a diffeomorphism and $\psi \circ f = \varphi$ (we can do this because f is a diffeomorphism so we can define the charts on N using the charts on M in this way), then $\psi_* \circ f_* = \varphi_*$ so it follows immediately from the fact that $\varphi_*[X, Y] = [\varphi_*X, \varphi_*Y]$ on $\mathbb{R}^n$ as we saw earlier. □

Example. Let (U, φ) be a chart on M. If ∂_i are the standard vector fields on $\mathbb{R}^n$, then

$$X_i = (\varphi^{-1})_*\partial_i$$

are vector fields on U and

$$[X_i, X_j] = (\varphi^{-1})_*[\partial_i, \partial_j] = 0.$$

We call these X_i the *coordinate vector fields* on U and will often use the notation X_i from now on to denote these coordinate vector fields.

Example. We see that if $f : \mathbb{R}^2 \to \mathcal{S}^2$ is our usual map given by $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ then $[\partial_\theta, \partial_\phi] = 0$ so

$$[f_*(\partial_\theta), f_*(\partial_\phi)] = 0.$$

Example. If $\varphi_S^{-1} : \mathbb{R}^3 \to \mathcal{S}^3 \setminus \{S\}$ then $Y_i = (\varphi_S^{-1})_*E_i$ from the earlier example satisfies $[Y_1, Y_2] = Y_3$ and cyclic permutations.

It is worth noting the following, as we will use it, though most of the time we only care about vector fields whose Lie bracket is zero.

Proposition 3.9. *The Lie bracket satisfies the Jacobi identity: i.e. if $X, Y, Z \in \Gamma(TM)$,*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Proof. **(Not examinable).** This can be done by direct computation in local coordinates, which we leave as an exercise. □

Remark. (Not examinable). Proposition 3.9 says that $\Gamma(TM)$ can actually be thought of as an infinite-dimensional Lie algebra.

3.6 Integral curves

We also want to think of vector fields of families of "arrows" on a manifold telling us which way to move. We do this by returning to the relationship between tangent vectors and curves in the manifold.

Given a curve $\alpha : (-\epsilon, \epsilon) \to M$, we can define $\alpha'(t) \in T_{\alpha(t)}M$ for all $t \in (-\epsilon, \epsilon)$ by $\alpha'(t) = \alpha_t'(0)$ where $\alpha_t(s) = \alpha(s+t)$. The map $t \mapsto \alpha'(t)$ from $(-\epsilon, \epsilon)$ into TM is smooth, so defines a vector field α' along α.

Let $X \in \Gamma(TM)$ and $p \in M$. There exists a unique curve $\alpha_p : (-\epsilon, \epsilon) \to M$ through p such that $\alpha_p'(t) = X(\alpha_p(t))$ for all $t \in (-\epsilon, \epsilon)$ since in a chart (U, φ) we can write $\varphi \circ \alpha(t) = (x_1(t), \ldots, x_n(t))$, $\varphi_*(X) = \sum_{i=1}^n a_i\partial_i$ and we have

$$\varphi_*(\alpha_p'(t)) = (\varphi \circ \alpha_p)'(t) = \sum_{i=1}^n x_i'(t)\partial_i$$

and

$$\varphi_*(X) = \sum_{i=1}^n a_i\partial_i,$$

so we have ODEs $x_i'(t) = a_i(x_1(t), \ldots, x_n(t))$ together with an initial condition $(x_1, \ldots, x_n)(0) = \varphi(p)$.

Definition 3.10. Given $X \in \Gamma(TM)$ and $p \in M$, by the discussion above there exists an open set $V \ni p$ such that for all $q \in V$ we have unique curves $\alpha_q : (-\epsilon, \epsilon) \to M$ such that $\alpha_q(0) = q$ and

$$\alpha_q'(t) = X(\alpha_q(t)).$$

These curves are called the *integral curves* of X (because we are essentially integrating the differential equation defined by the vector field X, which is a differential operator).

Example. For the vector fields ∂_i on $\mathbb{R}^n$ and $q \in \mathbb{R}^n$ the integral curve $\alpha_q(t) = (x_1(t), \ldots, x_n(t))$ with $\alpha_q(0) = q = (q_1, \ldots, q_n)$ is defined by

$$\sum_{j=1}^n x_j'(t)\partial_j = \partial_i$$

so

$$x_j' = \delta_{ij} \quad \Rightarrow \quad x_j(t) = q_j + \delta_{ij}t.$$

Hence the integral curves of ∂_i are straight lines $\alpha_q(t) = q + t\mathbf{e}_i$, where $\mathbf{e}_i$ is the unit vector in the x_i direction.

Example. If $X_j = -\sin\theta_j\partial_{2j-1} + \cos\theta_j\partial_{2j}$ are the standard vector fields on T^n and

$$\alpha(t) = (\cos\theta_1(t), \sin\theta_1(t), \ldots, \cos\theta_n(t), \sin\theta_n(t))$$

is the integral curve of X_j through $(\cos\theta_1, \sin\theta_1, \ldots, \cos\theta_n, \sin\theta_n)$, we see that

$$\sum_{i=1}^n \theta_i'(-\sin\theta_i\partial_{2i-1} + \cos\theta_i\partial_{2i}) = -\sin\theta_j\partial_{2j-1} + \cos\theta_j\partial_{2j},$$

so we have $\theta_i'(t) = \delta_{ij}$, which means that the integral curve is

$$\alpha(t) = (\cos\theta_1, \sin\theta_1, \ldots, \cos(\theta_j + t), \sin(\theta_j + t), \ldots, \cos\theta_n, \sin\theta_n).$$

Example. Let $X = x_1\partial_2 - x_2\partial_1$ and let $(a_1, a_2, a_3) \in \mathbb{R}^3$. The integral curve $\alpha(t) = (x_1(t), x_2(t), x_3(t))$ of X through x satisfies

$$x_1'(t)\partial_1 + x_2'(t)\partial_2 + x_3'(t)\partial_3 = x_1(t)\partial_2 - x_2(t)\partial_1.$$

Therefore,

$$x_1'(t) = -x_2(t), \quad x_2'(t) = x_1(t), \quad x_3'(t) = 0$$

which we can solve since $x_1''(t) = x_1(t)$ forces $x_1 = A\cos t + B\sin t$ and $x_2 = A\sin t - B\cos t$, which then means

$$x_1 = a_1\cos t - a_2\sin t,\ x_2 = a_2\cos t + a_1\sin t,\ x_3 = a_3.$$

Therefore the integral curves of X are circles in the plane where $x_3 = a_3$ is constant and centered at $(0, 0, a_3)$ with radius $\sqrt{a_1^2 + a_2^2}$.

Notice that X restricts to a vector field on the cylinder

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1,\ x_3 \in \mathbb{R}\}.$$

This is because by direct calculation you see that $T_{(x_1,x_2,x_3)}C \cong \mathrm{Span}\{(-x_2, x_1, 0), (0, 0, 1)\}$ since $C = F^{-1}(0)$ where $F(x_1, x_2, x_3) = x_1^2 + x_2^2 - 1$ and $\mathrm{d}F_{(x_1,x_2,x_3)} = (2x_1\ 2x_2\ 0)$. We see that the integral curves of X starting on C stay on C (as they must).

Example. It is straightforward to see that the integral curves of the vector fields $X_1 = f_*\partial_\theta$ and $X_2 = f_*\partial_\phi$ on $\mathcal{S}^2$, where $f : (0, \pi) \times (0, 2\pi) \to \mathcal{S}^2$ is our usual coordinates, are the lines of longitude and latitude respectively.

3.7 Flow

Observe that the map $(t,q)\mapsto\alpha_q(t)$ from $(-\epsilon,\epsilon)\times V$ into M is smooth by the theory of ODEs, so we can make the following definition.

Definition 3.11. Let $X\in\Gamma(TM)$ and $p\in M$. Let $V\ni p$ be an open set such that we have integral curves $\alpha_q:(-\epsilon,\epsilon)\to M$ of X through q for all $q\in V$.

We define the *flow* of X on V (or we say simply near p) as the family of smooth maps

$$\{\phi_t^X:V\to M\,:\,t\in(-\epsilon,\epsilon)\}$$

given by $\phi_t^X(q)=\alpha_q(t)$. Notice that ϕ_0^X is the identity on V.

The flow says how points on M move by the vector field X. Thus we can think of X as a family of "arrows" on M which point in the direction of the flow or, equivalently, the integral curves. The flow is only in general defined locally (in the sense that we cannot always take $V=M$), but quite often in examples we will see that the flow is globally defined.

Example. For ∂_i on $\mathbb{R}^n$ we saw that $\alpha_q(t)=q+t\mathbf{e}_i$ so $\phi_t^{\partial_i}(q)=q+t\mathbf{e}_i$, so the flow is just translation in the $\mathbf{e}_i$ direction.

Example. The flow of the vector field X_j on T^n is

$$\phi_t^{X_j}(\cos\theta_1,\sin\theta_1,\ldots,\cos\theta_n,\sin\theta_n)=(\cos\theta_1,\sin\theta_1,\ldots,\cos(\theta_j+t),\sin(\theta_j+t),\ldots,\cos\theta_n,\sin\theta_n).$$

Example. Recall the vector field X on the cylinder

$$C=\{(\cos\theta,\sin\theta,z)\in\mathbb{R}^3\,:\,\theta,z\in\mathbb{R}\}.$$

The flow of X on C is

$$\phi_t^X(\cos\theta,\sin\theta,z)=(\cos\theta\cos t-\sin\theta\sin t,\sin\theta\cos t+\cos\theta\sin t,z)=(\cos(\theta+t),\sin(\theta+t),z)$$

which are just rotations anti-clockwise around the circle with z fixed.

For another example, if we consider $Y=x_1\partial_2-x_2\partial_1+\partial_3$, which again is a vector field on C, we have that the integral curves now satisfy

$$x_1'(t)=-x_2(t),\quad x_2'(t)=x_1(t),\quad x_3'(t)=1,$$

so we have that the flow of Y is

$$\phi_t^Y(\cos\theta,\sin\theta,z)=(\cos(\theta+t),\sin(\theta+t),z+t),$$

which are "screw" motions around the cylinder.

Example. The flow of X_1 and X_2 (as we have previously studied) on $\mathcal{S}^2$ are the translations along the lines of longitude from the South pole to the North pole, and around the lines of latitude (anticlockwise) respectively.

We see that it is fairly straightforward to work out the flow of a vector field, which is one of the key ways to understand and visualise vector fields. The flow turns out to be very useful and has some important properties.

Proposition 3.12. *Let $p\in M$ and let $\{\phi_t^X:V\to M\,:\,t\in(-\epsilon,\epsilon)\}$ be the flow of $X\in\Gamma(TM)$ on $V\ni p$. Then $\phi_t^X\circ\phi_{t'}^X=\phi_{t+t'}^X$ if both sides are well-defined and ϕ_t^X is a local diffeomorphism at p.*

Proof. First, $\phi_t^X \circ \phi_{t'}^X(q) = \alpha_{\alpha_q(t')}(t)$ and $\phi_{t+t'}^X(q) = \alpha_q(t+t')$. Observe that α_q is the unique solution of $\alpha'(s) = X(\alpha(s))$ for $s \in (-\epsilon, \epsilon)$ with $\alpha(0) = q$, as well as the unique solution to the same differential equation with the condition $\alpha(t') = \alpha_q(t')$. However, the unique solution to this differential equation with this second initial condition is also $\alpha_{\alpha_q(t')}(s)$ by definition of the integral curve, so we have that $\alpha_q(t+t') = \alpha_{\alpha_q(t')}(t)$ and hence $\phi_t^X \circ \phi_{t'}^X = \phi_{t+t'}^X$ as claimed.

Suppose without loss of generality that $\phi_{-t}^X \circ \phi_t^X$ and $\phi_t^X \circ \phi_{-t}^X$ are well-defined, which will be true for ϵ small enough. Then $\phi_{-t}^X \circ \phi_t^X = \phi_0^X = \mathrm{id}$ and differentiating we see that

$$\mathrm{d}(\phi_{-t}^X)_{\phi_t^X(p)} \circ \mathrm{d}(\phi_t^X)_p = \mathrm{id}$$

so $\mathrm{d}(\phi_t^X)_p$ is an isomorphism and hence by Proposition 2.7 we see that ϕ_t^X is a local diffeomorphism. $\square$

Remark. (Not examinable). Proposition 3.12 says that the flow of a vector field defines a (local) one-parameter group of local diffeomorphisms. This shows that $T_{\mathrm{id}}\,\mathrm{Diff}(M)$, the tangent space to the identity of the diffeomorphism group, is $\Gamma(TM)$, the vector fields. Hence, the vector fields form the Lie algebra of the (infinite-dimensional) Lie group $\mathrm{Diff}(M)$.

3.8 Lie derivative

Let $X, Y \in \Gamma(TM)$, $p \in M$ and consider the flow ϕ_t^X of X near p so that we can look at how Y "changes" along the flow of X. First we can look at Y at time t along the integral curve α_p, which is the tangent vector $Y(\phi_t^X(p)) \in T_{\phi_t^X(p)}M$ (since $\phi_t^X(p) = \alpha_p(t)$). Second, since $\phi_{-t}^X \circ \phi_t^X = \mathrm{id}$, we see that $\phi_{-t}^X(\phi_t^X(p)) = p$ so

$$\mathrm{d}(\phi_{-t}^X)_{\phi_t^X(p)} : T_{\phi_t^X(p)}M \to T_pM.$$

In this way, we can map the tangent vector $Y(\phi_t^X(p))$ "back" into T_pM, i.e.

$$(\phi_{-t}^X)_*(Y(\phi_t^X(p)) \in T_pM.$$

Since this tangent vector lies now in T_pM we can compare this vector with the original tangent vector $Y(p)$ and in this way we can measure how Y "varies" in the direction given by X. This motivates our next important definition.

Definition 3.13. Given $X, Y \in \Gamma(TM)$ we define the *Lie derivative* of Y with respect to X by

$$\mathcal{L}_X Y(p) = \lim_{t\to 0} \frac{(\phi_{-t}^X)_*(Y(\phi_t^X(p)) - Y(p)}{t}$$

where $\{\phi_t^X \,:\, t \in (-\epsilon, \epsilon)\}$ is the flow of X near p.

The Lie derivative $\mathcal{L}_X Y$ is also a vector field on M since $\mathcal{L}_X Y(p) \in T_pM$ for all $p \in M$ and X, Y are smooth.

Example. Let $Y = \sum_{j=1}^n b_j \partial_j$ be a vector field on $\mathbb{R}^n$. We know that $\phi_t^{\partial_i}(p) = p + t\mathbf{e}_i$ so $(\phi_{-t}^{\partial_i})_* = \mathrm{id}$ (once we identify the tangent spaces to $T_{p+t\mathbf{e}_i}\mathbb{R}^n$ and $T_p\mathbb{R}^n$). Hence,

$$\begin{aligned}\mathcal{L}_{\partial_i} Y(p) = \lim_{t\to 0} \frac{(\phi_{-t}^{\partial_i})_*(Y(\phi_t^{\partial_i}(p)) - Y(p)}{t} &= \lim_{t\to 0} \frac{\sum_{j=1}^n b_j(p+t\mathbf{e}_i)(\phi_{-t}^{\partial_i})_*\partial_j - b_j(p)\partial_j}{t} \\ &= \sum_{j=1}^n \lim_{t\to 0} \frac{b_j(p+t\mathbf{e}_i) - b_j(p)}{t}\partial_j = \sum_{j=1}^n \frac{\partial b_j}{\partial x_i}(p)\partial_j.\end{aligned}$$

In particular,

$$\mathcal{L}_{\partial_i}\partial_j = 0$$

and if $X = x_1\partial_2 - x_2\partial_1$ then

$$\mathcal{L}_{\partial_1}X = \partial_2, \quad \mathcal{L}_{\partial_2}X = -\partial_1, \quad \mathcal{L}_{\partial_3}X = 0.$$

Computing the Lie derivative seems very difficult! However, we have the following invaluable result, which also explains the name Lie bracket.

Proposition 3.14. $\mathcal{L}_X Y = [X, Y]$.

We shall not prove this because the proof is long and uninformative, and is proved just in local coordinates. It implies that

$$\mathcal{L}_Y X = -\mathcal{L}_X Y, \quad \mathcal{L}_X(Y+Z) = \mathcal{L}_X Y + \mathcal{L}_X Z \quad \text{and} \quad \mathcal{L}_X(fY) = f\mathcal{L}_X Y + X(f)Y.$$

Notice however that although $\mathcal{L}_{X+Y} Z = \mathcal{L}_X Z + \mathcal{L}_Y Z$ we have

$$\mathcal{L}_{fX} Y = -\mathcal{L}_Y(fX) = f\mathcal{L}_X Y - Y(f)X,$$

so one cannot simply "pull out" the function f from in front of X.

Remark. (Not examinable). The observations above say that the Lie derivative *does not* define a connection on the tangent bundle.

Example. Let $X = \sum_{i=1}^n a_i \partial_i$, then

$$\mathcal{L}_X \partial_j = \sum_{i=1}^n a_i \mathcal{L}_{\partial_i} \partial_j - \sum_{i=1}^n \frac{\partial a_i}{\partial x_j} \partial_i = -\sum_{i=1}^n \frac{\partial a_i}{\partial x_j} \partial_i.$$

In particular, if $X = x_1 \partial_2 - x_2 \partial_1$ then

$$\mathcal{L}_X \partial_1 = -\partial_2, \quad \mathcal{L}_X \partial_2 = \partial_1, \quad \mathcal{L}_X \partial_3 = 0.$$

Example. Let $X_1 = x_3\partial_2 - x_2\partial_3$, $X_2 = x_1\partial_3 - x_3\partial_1$ and $X_3 = x_2\partial_1 - x_1\partial_2$ on $\mathbb{R}^3$. Then $\mathcal{L}_{X_1} X_2 = [X_1, X_2] = X_3$.

Example. If (U, φ) is a chart on M and ∂_i are standard vector fields on $\mathbb{R}^n$ and $X_i = (\varphi^{-1})_* \partial_i$ are the coordinate vector fields then $\mathcal{L}_{X_i} X_j = [X_i, X_j] = 0$.

4 Differential forms

In this section we want to study objects which are "dual" to vector fields, called 1-forms, and more generally differential forms. Differential forms enable us to study many important issues on manifolds, including orientability and integration. This will play a key role when we study the invariants of manifolds known as de Rham cohomology later.

4.1 Vector bundles

The tangent bundle is a key example of a more general object called a vector bundle, which appears a lot in geometry, in particular in the study of differential forms.

Definition 4.1. A manifold E is a *vector bundle* over a manifold M if

- there exists a smooth surjective map $\pi : E \to M$ such that
- $\pi^{-1}(p)$ is a vector space for all $p \in M$ and
- for all $p \in M$ there exists an open set $U \ni p$ and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ such that $\psi : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^m$ is an isomorphism for all $q \in U$.

The integer m is the same for all $p \in M$ and is called the *rank* of the vector bundle. Clearly if M is n-dimensional then E is $(m+n)$-dimensional. We often call E the *total space* and M the *base* of the vector bundle.

Example. Given any manifold M we always have the obvious vector bundle $M \times \mathbb{R}^m$. The simplest example is the cylinder $\mathcal{S}^1 \times \mathbb{R} \cong \{(\cos\theta, \sin\theta, z) \,:\, \theta, z \in \mathbb{R}\}$.

Example. The tangent bundle is a vector bundle of rank n over an n-dimensional manifold.

We have a natural map $\pi : E \to M$ for a vector bundle but what about maps from M to E? These maps play an important role in geometry.

Definition 4.2. Let E be a vector bundle over M. A *section* of E is a smooth map $s : M \to E$ such that $(\pi \circ s)(p) = p$ for all $p \in M$. We denote the set of sections of E by $\Gamma(E)$, which is naturally a vector space because $s(p) \in \pi^{-1}(p)$, which is a vector space for all $p \in M$.

Example. A section of TM is a vector field.

Remark. The graph of a section $\{(p, s(p)) : p \in M\}$ is clearly diffeomorphic to M using the projection map π, so since we can think of E locally as a cylinder over M we can think of s as a "cross-section" of the cylinder, which is where the name comes from.

Example. If we look at the cylinder $C = \mathcal{S}^1 \times \mathbb{R}$, we see that we always have obvious sections $s : \mathcal{S}^1 \to C$ given by $s(\cos\theta, \sin\theta) = (\cos\theta, \sin\theta, z)$ for any $z \in \mathbb{R}$ (which is a "horizontal" circle). However, we also have more interesting sections such as $s(\cos\theta, \sin\theta) = (\cos\theta, \sin\theta, \cos\theta)$ (which is a "sloped" circle).

Example. If we let

$$S^2T_p^*M = \{\text{symmetric bilinear maps } g_p : T_pM \times T_pM \to \mathbb{R}\}$$

then

$$S^2T^*M = \cup_{p\in M} S^2T_p^*M$$

is a vector bundle of rank $\frac{1}{2}n(n+1)$ over an n-dimensional manifold M.

A *Riemannian metric* g on M is a section of S^2T^*M, i.e. $g \in \Gamma(S^2T^*M)$, which is positive definite (meaning that g_p is positive definite for all $p \in M$). We see these again at the end of the course.

We will see some other useful examples of vector bundles shortly. Given a vector bundle, we want to know if it is interesting or not and one way to do this is to see if it is a product or not.

Definition 4.3. A vector bundle E of rank m over M is *trivial* if there exists a diffeomorphism $\psi : E \to M \times \mathbb{R}^m$ such that $\psi : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^m$ is an isomorphism for all $p \in M$; i.e. a bundle isomorphism between E and the trivial bundle $M \times \mathbb{R}^m$.

It is not always clear if a bundle is trivial or not but we have a nice test by looking at the sections.

Proposition 4.4. *A vector bundle of rank m is trivial if and only if it has m linearly independent sections.*

Proof. If E is trivial we have a diffeomorphism $\chi : M \times \mathbb{R}^m \to E$ so that $\chi : \{p\} \times \mathbb{R}^m \to \pi^{-1}(p)$ is an isomorphism for all $p \in M$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be a basis for $\mathbb{R}^m$. We can then define maps $s_i : M \to E$ for $i = 1, \ldots, m$ by $s_i(p) = \chi(p, \mathbf{e}_i)$. Clearly $\pi \circ s_i(p) = p$ (where $\pi : E \to M$ is the natural projection) for all $p \in M$ and s_i is smooth because χ is smooth, so $s_i \in \Gamma(E)$. Moreover, if

$$(\lambda_1 s_1 + \ldots + \lambda_m s_m)(p) = 0$$

then χ defines an isomorphism between $\{p\} \times \mathbb{R}^m$ and $\pi^{-1}(p)$ so we have that

$$0 = \lambda_1 \chi(p, \mathbf{e}_1) + \ldots + \lambda_m \chi(p, \mathbf{e}_m) = \chi(p, \lambda_1 \mathbf{e}_1 + \ldots + \lambda_m \mathbf{e}_m),$$

which forces $\lambda_1 = \ldots = \lambda_m = 0$ as the $\mathbf{e}_i$ are linearly independent. Hence the sections s_i are everywhere linearly independent.

Suppose instead that we have linearly independent $s_i \in \Gamma(E)$ for $i = 1, \ldots, m$. Since the $\mathbf{e}_i$ form a basis for $\mathbb{R}^m$ we can define $\chi : M \times \mathbb{R}^m \to E$ by

$$\chi(p, \lambda_1 \mathbf{e}_1 + \ldots + \lambda_m \mathbf{e}_m) = \lambda_1 s_1(p) + \ldots + \lambda_m s_m(p).$$

Clearly $\chi : \{p\} \times \mathbb{R}^m \to \pi^{-1}(p)$ is a well-defined isomorphism and $\pi \circ \chi(p, \mathbf{x}) = p$ so χ is a bijection. Clearly, χ is smooth and its inverse is smooth so χ gives the required bundle isomorphism. □

4.2 Exterior algebra

In this section we take a short interlude to introduce the algebra necessary to discuss differential forms. Recall that if V is a vector space (over $\mathbb{R}$), which we fix for the duration of this section, then V^* denotes its dual space, i.e. the space of linear maps from V to $\mathbb{R}$. For the study of forms on a manifold M, the relevant vector spaces V will just be the tangent spaces T_pM.

Definition 4.5. Given a vector space V we can define the *tensor product* $\otimes^k V^*$ to be the set of multilinear maps $T : V \times \ldots \times V \to \mathbb{R}$ (so linear on each entry) which act on k-tuples of vectors in V. We call elements of $\otimes^k V^*$ (covariant) k-tensors.

Example. Since the elements of V^* are linear maps, we can define an element of V^* by prescribing its action on a basis for V. In particular, given a basis $\{X_1, \ldots, X_n\}$ for V we can define a basis $\{\xi_1, \ldots, \xi_n\}$ for V^* by the requirement that $\xi_i(X_j) = \delta_{ij}$.

We have a useful operation on (covariant) tensors known as *tensor product*: if $S \in \otimes^k V^*$ and $T \in \otimes^l V^*$ we have $S \otimes T \in \otimes^{k+l} V^*$ given by

$$(S \otimes T)(X_1, \ldots, X_{k+l}) = S(X_1, \ldots, X_k) T(X_{k+1}, \ldots, X_{k+l}).$$

Example. We see that if $\{\xi_1, \ldots, \xi_n\}$ is a basis for V^*, then the tensor products

$$\xi_{i_1} \otimes \ldots \otimes \xi_{i_k}$$

for $1 \leq i_1, \ldots, i_k \leq n$ form a basis for $\otimes^k V^*$. It is important to realise that not every element of $\otimes^k V^*$ need be a k-fold tensor product, but only a linear combination of such tensor products. For example, elements in $\otimes^2 V^*$ need not necessarily be of the form $\xi_1 \otimes \xi_2$: e.g. if $V^* = \mathrm{Span}\{\xi_1, \xi_2\}$ is 2-dimensional then

$$\otimes^2 V^* = \mathrm{Span}\{\xi_1 \otimes \xi_1, \xi_1 \otimes \xi_2, \xi_2 \otimes \xi_1, \xi_2 \otimes \xi_2\} \cong M_2(\mathbb{R}).$$

We now want to define the spaces of *symmetric* tensors $S^k V^*$ and *alternating* tensors $\Lambda^k V^*$, and it is the latter which will be of primary interest for differential forms. Specifically, for $\sigma \in S_k$ (the permutation group of $\{1, \ldots, k\}$), we want for $S \in S^k V^*$ and $T \in \Lambda^k V^*$ that

$$S(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = S(X_1, \ldots, X_k) \quad \text{and} \quad T(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = \mathrm{sign}(\sigma) T(X_1, \ldots, X_k).$$

To achieve this we have a symmetrization map $\mathcal{S}$ and an alternating map $\mathcal{A}$ acting on $\otimes^k V^*$ defined by:

$$\mathcal{S}T(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} T(X_{\sigma(1)}, \ldots, X_{\sigma(k)})$$

and

$$\mathcal{A}T(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) T(X_{\sigma(1)}, \ldots, X_{\sigma(k)}).$$

We can then make our definition.

Definition 4.6. We define $S^k V^*$ and $\Lambda^k V^*$ as the images of $\mathcal{S}$ and $\mathcal{A}$ acting on $\otimes^k V^*$, respectively. If $T \in S^k V^*$ (i.e. T is a symmetric k-tensor) then $\mathcal{S}T = T$ and if $T \in \Lambda^k V^*$ (i.e. T is an alternating k-tensor) then $\mathcal{A}T = T$.

One calls $\Lambda^* V^* = \oplus_{k=0}^{\infty} (\Lambda^k V^*)$ the *exterior algebra* (of V^*). Notice that $\Lambda^k V^* = \{0\}$ for k larger than the dimension of V.

Remark. (Not examinable). By taking dual spaces one can define $\otimes^k V$, $S^k V$ and $\Lambda^k V$, and one can also define objects such as $V \otimes W$, but we shall not require them in this course.

Example. By definition, we take $\Lambda^0 V^* = \mathbb{R}$, and we take $\Lambda^1 V^* = V^*$, so just the dual space of V.

Example. If $g \in S^2 V^*$ then $g(X, Y) = g(Y, X)$ for all $X, Y \in V$ and g is bilinear, so linear in both entries. For example, if $X, Y \in \mathbb{R}^n$ then we can define $g_0 \in S^2(\mathbb{R}^n)^*$ by

$$g_0(X, Y) = \langle X, Y \rangle,$$

i.e. the usual inner product on $\mathbb{R}^n$.

If $\omega \in \Lambda^2 V^*$ then $\omega(X, Y) = -\omega(Y, X)$ (so $\omega(X, X) = 0$) for all $X, Y \in V$ and is bilinear. For example, if $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$ we can define $\omega_0 \in \Lambda^2(\mathbb{R}^2)^*$ by

$$\omega_0((u_1, u_2), (v_1, v_2)) = u_1 v_2 - u_2 v_1.$$

Notice that this is nothing but the determinant of the matrix whose rows (or columns) are the vectors $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$.

The last thing we need to understand in this short algebraic interlude is how to obtain a basis for $\Lambda^k V^*$. This is achieved using the *wedge product*.

Definition 4.7. If $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$ then the wedge product $\omega \wedge \eta \in \Lambda^{k+l} V^*$ is defined by

$$\omega \wedge \eta = \frac{(k+l)!}{k! l!} \mathcal{A}(\omega \otimes \eta).$$

It is important to note that

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta.$$

In particular, if $\xi \in V^*$ then $\xi \wedge \xi = 0$.

Example. Given a basis $\{\xi_1, \ldots, \xi_n\}$ for V^*, the wedge products

$$\xi_{i_1} \wedge \ldots \wedge \xi_{i_k}$$

for $1 \leq i_1 < \ldots < i_k \leq n$ form a basis for $\Lambda^k V^*$: such elements of $\Lambda^k V^*$ are called *decomposable*. Again, it is important to note that not every element of $\Lambda^k V^*$ need be decomposable, but just a sum of decomposables. The elements of $\Lambda^k V^*$ which are not decomposable are called indecomposable.

Example. If V^* has basis $\{\xi_1, \xi_2\}$ then $\Lambda^2 V^*$ is spanned by $\xi_1 \wedge \xi_2$ and thus is 1-dimensional.

In general, if V^* is n-dimensional with basis $\{\xi_1, \ldots, \xi_n\}$ then $\Lambda^n V^*$ is 1-dimensional and spanned by the (decomposable) element $\xi_1 \wedge \ldots \wedge \xi_n$.

Example. Elements of V^* are always trivially decomposable. It turns out that $\Lambda^2 V^*$, where V has dimension at least 4, must contain indecomposable elements.

4.3 Forms on manifolds

We can now define forms on manifolds: just as for manifolds we drop the adjective "differential" for convenience.

Definition 4.8. We let

$$\Lambda^k T^* M = \cup_{p \in M} \Lambda^k T_p^* M,$$

which is a vector bundle over M. The sections $\Gamma(\Lambda^k T^* M)$ of $\Lambda^k T^* M$ are called k-forms.

Example. The 0-forms are the functions $f : M \to \mathbb{R}$.

Example. Recall that $\Lambda^1 T_p^* M = T_p^* M$ and so we wrtie $\Lambda^1 T^* M = T^* M$. $T^* M$ is a rank n vector bundle over an n-dimensional manifold M called the cotangent bundle, and $T_p^* M$ is called the cotangent space to M at p.

Example. $\Lambda^n T^* M$ is a rank 1 vector bundle over an n-dimensional manifold M.

Suppose $\xi \in \Gamma(T^* M)$ and $X \in \Gamma(TM)$. Then $\xi(p) \in T_p^* M$ so $\xi(p) : T_p M \to \mathbb{R}$. Hence $\xi(p)(X(p)) \in \mathbb{R}$ for all $p \in M$. Hence $\xi(X) : M \to \mathbb{R}$ is a smooth function. Thus 1-forms are "dual" to vector fields.

Example. If TM is trivial we have n linearly independent vector fields $X_1, \ldots, X_n$ by Proposition 3.5, so we can define $\xi_1, \ldots, \xi_n$ by $\xi_i(X_j) = \delta_{ij}$, where δ_{ij} is the function which is 1 if $i = j$ and 0 otherwise. Then $\xi_1, \ldots, \xi_n$ are linearly independent so by Proposition 4.4 we see that $T^* M$ is trivial as well.

Since we have the standard vector fields $\partial_1 = \frac{\partial}{\partial x_1}, \ldots, \partial_n = \frac{\partial}{\partial x_n}$ on $\mathbb{R}^n$ which trivialize $T\mathbb{R}^n$, we have the corresponding 1-forms

$$\mathrm{d}x_1, \ldots, \mathrm{d}x_n$$

which are defined, as in the previous example, by $\mathrm{d}x_i(\partial_j) = \delta_{ij}$ and the 1-forms are linear maps so

$$\mathrm{d}x_i\Big(\sum_{j=1}^n b_j \partial_j\Big) = b_i.$$

for functions $b_1, \ldots, b_n$. We also have that every 1-form on $\mathbb{R}^n$ can be written as a combination of these 1-forms $\sum_{i=1}^n a_i \mathrm{d}x_i$ for functions a_i and we have a basis for the k-forms on $\mathbb{R}^n$ by taking the wedge products of 1-forms:

$$\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$

for $1 \leq i_1 < \ldots < i_k \leq n$. In particular, any n-form on $\mathbb{R}^n$ must be a multiple of

$$\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.$$

As for vector fields, it is important to keep in mind the examples of forms on $\mathbb{R}^n$ since they give local models for all forms, as we shall see explicitly in this next section. We will study a couple of examples in detail, though we will typically be able to do these calculations much faster once we get used to how they work.

Example. It is clear that if we have a manifold $M \subseteq \mathbb{R}^n$ then we can restrict a form on $\mathbb{R}^n$ to M by only acting on tangent vectors to M. For example, if we consider the 1-form

$$\xi = \frac{x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1}{x_1^2 + x_2^2}$$

on $\mathbb{R}^2 \setminus \{0\}$ then we can evaluate it on $x_1\partial_2 - x_2\partial_1$ and we see that

$$\begin{aligned} \xi(x_1\partial_2 - x_2\partial_1) &= \frac{x_1 \mathrm{d}x_2(x_1\partial_2 - x_2\partial_1) - x_2 \mathrm{d}x_1(x_1\partial_2 - x_2\partial_1)}{x_1^2 + x_2^2} \\ &= \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} = 1. \end{aligned}$$

Whereas

$$\xi(x_1\partial_1 + x_2\partial_2) = \frac{-x_1x_2 + x_1x_2}{x_1^2 + x_2^2} = 0.$$

Example. Given vector fields $X = u_1\partial_1 + u_2\partial_2$ and $Y = v_1\partial_1 + v_2\partial_2$ on $\mathbb{R}^2$ we can calculate

$$\begin{aligned} \mathrm{d}x_1 \wedge \mathrm{d}x_2(X, Y) &= \frac{(1+1)!}{1!1!} \frac{1}{2!}(\mathrm{d}x_1 \otimes \mathrm{d}x_2(X, Y) - \mathrm{d}x_2 \otimes \mathrm{d}x_1(X, Y) \\ &= \mathrm{d}x_1(X)\mathrm{d}x_2(Y) - \mathrm{d}x_2(X)\mathrm{d}x_1(Y). \end{aligned}$$

Now, since

$$\mathrm{d}x_j(X) = u_j \quad \text{and} \quad \mathrm{d}x_j(Y) = v_j$$

we see that

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2(X, Y) = u_1v_2 - u_2v_1.$$

Hence, $\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ can be identified with the alternating 2-tensor we called ω_0 before.

Example. Given a smooth function $f : M \to \mathbb{R}$ we can define a 1-form $\mathrm{d}f$ on M by

$$\mathrm{d}f(p)(X) = \mathrm{d}f_p(X)$$

for $p \in M$ and $X \in T_pM$. This 1-form (unfortunately) is often called the differential of f.

If $f : \mathbb{R}^n \to \mathbb{R}$ then

$$\mathrm{d}f(p)(\partial_i) = \frac{\partial f}{\partial x_i}(p).$$

Hence

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

Therefore, $\mathrm{d}f = 0$ if and only if f is constant.

Since functions f on M can be viewed locally as functions on $\mathbb{R}^n$, we see that $\mathrm{d}f = 0$ if and only if f is locally constant: i.e. constant on connected components of M.

4.4 Pullback

Now, just as the differential of a smooth map between manifolds gave a natural map between tangent spaces (and thus vector fields) we also get a natural map between cotangent spaces (and thus forms), but it goes in the opposite direction.

Suppose $f : M \to N$ is a smooth map. Then $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is well-defined. Now given an element $\eta \in T^*_{f(p)}N$, say we want to define ω in T^*_pM, so it has to act on some $X \in T_pM$. But the obvious way to define this ω is to pushforward X to N and act on it with η, i.e. $\omega(X) = \eta(\mathrm{d}f_p(X))$. This is exactly what we do. Unlike for vector fields we can define the pullback on forms for any smooth map.

Definition 4.9. Let $f : M \to N$ be a smooth map. If $\omega \in \Gamma(\Lambda^k T^*N)$ we can define the *pullback* $f^*\omega \in \Gamma(\Lambda^k T^*M)$ by

$$(f^*\omega)(p)(X_1, \ldots, X_k) = \omega(f(p))(\mathrm{d}f_p(X_1), \ldots, \mathrm{d}f_p(X_k))$$

for all $p \in M$ and $X \in T_pM$.

Remark. Notice that $(f \circ g)^* = g^* \circ f^*$ and $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Example. As before define a 1-form on $\mathbb{R}^2 \setminus \{(0, 0)\}$ by

$$\xi = \frac{x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1}{x_1^2 + x_2^2}$$

and we saw that if $X = x_1\partial_2 - x_2\partial_1$ then $\xi(X) = 1$. If we let $f : \mathbb{R} \to \mathbb{R}^2$ be $f(\theta) = (\cos\theta, \sin\theta)$ then $f(\mathbb{R}) = \mathcal{S}^1$ and

$$f_*(\partial_\theta) = -\sin\theta\partial_1 + \cos\theta\partial_2$$

is the restriction of X to $\mathcal{S}^1$. Therefore

$$f^*\xi(\partial_\theta) = \xi(f_*\partial_\theta) = \xi(X) = 1,$$

so $f^*\xi = \mathrm{d}\theta$, the 1-form dual to ∂_θ.

Given any chart (U, φ) on M we know that $\varphi^{-1} : \varphi(U) \to U$ so if ω is a k-form on U then $(\varphi^{-1})^*\omega$ is a k-form on $\varphi(U) \subseteq \mathbb{R}^n$. Hence any k-form on a manifold can always be locally viewed as k-form on M.

We now describe the important example of pullback by a linear map.

Example. Let $A = (a_{ij}) \in M_n(\mathbb{R})$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map $f(x) = Ax$. Since $\mathrm{d}f_x = A$ for all $x \in \mathbb{R}^n$ we see that

$$f_*\partial_i = \sum_{j=1}^n a_{ji}\partial_j$$

(i.e. the image of ∂_i under f_* corresponds to the ith column of A), which means that we can identify f_* with the linear map given by A.

We then see that

$$(f^*\mathrm{d}x_i)(\partial_j) = \mathrm{d}x_i(f_*\partial_j) = \mathrm{d}x_i(\sum_{k=1}^n a_{kj}\partial_k) = a_{ij}.$$

Therefore,

$$f^*\mathrm{d}x_i = \sum_{j=1}^n a_{ij}\mathrm{d}x_j,$$

which means that we can identify f^* with the linear map given by A^{T}.

It is also straightforward to see (and you should check!), using the relationship between pullback and wedge product, that if $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ then

$$f^*\Omega_0 = \det A\,\Omega_0.$$

In particular, f preserves Ω_0 if and only if $A \in \mathrm{SL}(n, \mathbb{R})$.

Example. The previous example easily generalises to show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is any smooth map and $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ then

$$f^*\Omega_0 = \det(f_*)\,\Omega_0.$$

This will be important later.

4.5 Exterior derivative

We now want to define a linear map which takes k-forms to $k+1$-forms called the *exterior derivative* d. We start with the definition on $\mathbb{R}^n$ and then generalize to manifolds. We will see on $\mathbb{R}^3$ that it is intimately related to the familiar div, grad and curl. This will be important when we come to integration later.

Definition 4.10. We define the *exterior derivative* of a decomposable k-form on $\mathbb{R}^n$ by

$$\mathrm{d}(a\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}) = \sum_{j=1}^{n} \frac{\partial a}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k},$$

(here $a : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary function) and then extend the exterior derivative to all k-forms on $\mathbb{R}^n$ by linearity.

Example. In the special case of functions $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \mathrm{d}x_j,$$

which is nothing other than the same 1-form $\mathrm{d}f$ we saw earlier.

Notice for a function $f : \mathbb{R}^3 \to \mathbb{R}$, using the basis $\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$ for the 1-forms on $\mathbb{R}^3$, we can identify $\mathrm{d}f$ with

$$\mathrm{grad} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right).$$

We see from the definition that

$$\begin{aligned} \mathrm{d} \circ \mathrm{d}(a\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}) &= \mathrm{d} \left(\sum_{j=1}^{n} \frac{\partial a}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} \right) \\ &= \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 a}{\partial x_j \partial x_l} \mathrm{d}x_l \wedge \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} = 0, \end{aligned}$$

since the second order partial derivative is symmetric in j and l, whilst the wedge product is skew in j and l. We deduce the important fact that

$$\mathrm{d}^2 = 0.$$

Example. For a 1-form on $\mathbb{R}^n$ we have

$$\mathrm{d}(\sum_{i=1}^{n} a_i \mathrm{d}x_i) = \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_i$$

In particular, on $\mathbb{R}^3$ we have

$$\mathrm{d}(a_1\mathrm{d}x_1 + a_2\mathrm{d}x_2 + a_3\mathrm{d}x_3) = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right) \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \right) \mathrm{d}x_3 \wedge \mathrm{d}x_1 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) \mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

If we view the 1-form as the vector-valued function $F = (a_1, a_2, a_3) : \mathbb{R}^3 \to \mathbb{R}^3$, then by choosing the basis $\{\mathrm{d}x_2 \wedge \mathrm{d}x_3, \mathrm{d}x_3 \wedge \mathrm{d}x_1, \mathrm{d}x_1 \wedge \mathrm{d}x_2\}$ for the 2-forms on $\mathbb{R}^3$, we can identify the exterior derivative of the 1-form with

$$\operatorname{curl} F = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}, \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}, \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right).$$

The statement $\mathrm{d}^2 f = 0$ for $f : \mathbb{R}^3 \to \mathbb{R}$ can therefore be re-interpreted as

$$\operatorname{curl}\operatorname{grad} f = 0.$$

Example. For a 2-form on $\mathbb{R}^3$ we have

$$\mathrm{d}\left(a_1\mathrm{d}x_2 \wedge \mathrm{d}x_3 + a_2\mathrm{d}x_3 \wedge \mathrm{d}x_1 + a_3\mathrm{d}x_1 \wedge \mathrm{d}x_2\right) = \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3.$$

Therefore, taking the obvious bases of 2-forms and 3-forms on $\mathbb{R}^3$, we may view the 2-form as the vector-valued function $F = (a_1, a_2, a_3)$ as in the previous example, but now the exterior derivative is interpreted as

$$\operatorname{div} F = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}.$$

The statement $\mathrm{d}^2\omega = 0$ for a 1-form ω on $\mathbb{R}^3$ can thus be interpreted as the familiar statement

$$\operatorname{div}\operatorname{curl} F = 0.$$

Example. For an n-form Ω on $\mathbb{R}^n$ we always have $\mathrm{d}\Omega = 0$ since it would be an $n+1$-form which is zero automatically.

Example. Let

$$\xi = \frac{x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1}{x_1^2 + x_2^2}$$

be a 1-form on $\mathbb{R}^2 \setminus \{0\}$. Then

$$\mathrm{d}\xi = \partial_1\left(\frac{x_1}{x_1^2+x_2^2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \partial_2\left(\frac{x_2}{x_1^2+x_2^2}\right)\mathrm{d}x_2 \wedge \mathrm{d}x_1 = \frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2}(\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_2 \wedge \mathrm{d}x_1) = 0$$

since $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = -\mathrm{d}x_2 \wedge \mathrm{d}x_1$.

Another important property of the exterior derivative, which is immediate from the definition and some elementary algebra using the wedge product, is that if ω is a k-form and η is an l-form then

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k\omega \wedge \mathrm{d}\eta.$$

A final observation we will make concerning forms on $\mathbb{R}^n$ is that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map and ω is a k-form on $\mathbb{R}^m$ then

$$\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega).$$

This follows from the relationship between pullback and wedge product, which enables one to prove the result by induction starting from the case of 0-forms (i.e. functions), where the statement is essentially the Chain rule.

Definition 4.11. We define a linear map $\mathrm{d} : \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{k+1}T^*M)$ called the *exterior derivative* by requiring that if $\omega \in \Gamma(\Lambda^k T^*M)$ and (U, φ) is a chart then

$$\mathrm{d}\omega|_U = \varphi^*(\mathrm{d}(\varphi^{-1})^*\omega|_U);$$

i.e. we define it using the formula on Euclidean space using the coordinates given by the chart. Using the properties of d on Euclidean space, this is not hard to check that this is well-defined (i.e. it agrees on overlaps of charts).

The exterior derivative has the following properties (which follow from having these properties on Euclidean space).

- For a function $f : M \to \mathbb{R}$, $\mathrm{d}f$ is the 1-form called the differential of f we saw earlier.
- $\mathrm{d}^2 = 0$, i.e. $\mathrm{d}(\mathrm{d}\omega) = 0$ for all forms ω.
- If ω is a k-form and η is an l-form then $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$.
- If $f : M \to N$ is a smooth map and ω is a form on N then $\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)$.

Remark. (Not examinable). The exterior derivative is in fact characterized by the first three properties listed above.

Definition 4.12. We can define *closed* forms ω by the condition that $\mathrm{d}\omega = 0$, and *exact* forms ω by the condition that $\omega = \mathrm{d}\eta$.

Notice that $\mathrm{d}^2 = 0$ implies that exact forms are closed, but the converse is not true in general: this will be important later.

Our next example is very important in differential geometry.

Example. Let $\pi : T^*M \to M$ be the projection. If $(x, \eta) \in T^*M$ then

$$\mathrm{d}\pi_{(x,\eta)} : T_{(x,\eta)}T^*M \to T_xM \text{ so } \mathrm{d}\pi^*_{(x,\eta)} : T^*_xM \to T^*_{(x,\eta)}T^*M.$$

Hence, for $\xi \in T^*_xM$,

$$\mathrm{d}\pi^*_{(x,\eta)}\xi \in T^*_{(x,\eta)}T^*M$$

is a cotangent vector on T^*M. In particular,

$$\mathrm{d}\pi^*_{(x,\xi)}\xi \in T^*_{(x,\xi)}T^*M,$$

so we can define the canonical 1-form τ on T^*M by

$$\tau(x, \xi) = \mathrm{d}\pi^*_{(x,\xi)}\xi.$$

In local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on T^*M, τ is given by $\sum_{i=1}^n p_i \mathrm{d}q_i$.

We then let $\omega = -\mathrm{d}\tau$. Clearly ω is closed (as it is exact) and in local coordinates ω is given by

$$-\mathrm{d}\left(\sum_{i=1}^n p_i \mathrm{d}q_i\right) = \sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

Thus $\omega^n = \omega \wedge \ldots \wedge \omega$ is given locally by (a constant multiple of) $\mathrm{d}q_1 \wedge \ldots \wedge \mathrm{d}q_n \wedge \mathrm{d}p_1 \wedge \ldots \wedge \mathrm{d}p_n$ which is nowhere vanishing (we call ω nondegenerate).

Hence the 2-form ω is a *symplectic form* and T^*M is a *symplectic manifold*.

Remark. (Not examinable). Symplectic manifolds play an important role in geometry and topology and in mechanics where the coordinates q_i are thought of as position and p_i as momentum. Darboux's theorem states that all symplectic manifolds locally look like $\mathbb{R}^n \times \mathbb{R}^n$ with the symplectic form $\sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i$.

4.6 Lie derivative and Cartan's formula

We have another way to differentiate forms ω and that it is by using a vector field X. This will be the analogue of the Lie derivative we saw before. Notice that the flow ϕ_t^X of X sends p to $\phi_t^X(p)$ so the pullback acts in the reverse direction

$$(\phi_t^X)^* : T^*_{\phi_t^X(p)}M \to T^*_pM.$$

So, if ω is a 1-form, we can compare $\omega(\phi_t^X(p)) \in T^*_{\phi_t^X(p)}M$ to $\omega(p) \in T^*_pM$ by using this pullback, which then motivates our definition of the Lie derivative.

Definition 4.13. Given $X \in \Gamma(TM)$ and $\omega \in \Gamma(\Lambda^k T^*M)$, the *Lie derivative* of ω with respect to X is given by

$$\mathcal{L}_X\omega(p) = \lim_{t\to 0} \frac{(\phi_t^X)^*\big(\omega(\phi_t^X(p))\big) - \omega(p)}{t}.$$

where $\{\phi_t^X : t \in (-\epsilon, \epsilon)\}$ is the flow of X near p.

The Lie derivative $\mathcal{L}_X\omega$ is also a k-form on M since $\mathcal{L}_X\omega(p) \in \Lambda^k T_p^*M$ and X, ω are smooth.

Example. In the case of a smooth function $f : M \to \mathbb{R}$, a 0-form, then

$$\mathcal{L}_X f = X(f).$$

Example. (Not examinable). On $\mathbb{R}^n$,

$$\begin{aligned}\mathcal{L}_X \mathrm{d}x_j(\partial_k) &= \lim_{t\to 0} \frac{(\phi_t^X)^*\mathrm{d}x_j(\partial_k) - \mathrm{d}x_j(\partial_k)}{t} = \lim_{t\to 0} \frac{\mathrm{d}x_j((\phi_t^X)_*\partial_k - \partial_k)}{t} \\ &= \mathrm{d}x_j\left(\lim_{t\to 0} \frac{(\phi_{-t}^X)_*\partial_k - \partial_k}{-t}\right) = -\mathrm{d}x_j(\mathcal{L}_X\partial_k)\end{aligned}$$

by Definition 3.13 (noticing that we had to change t to $-t$ in the limit which gave us the sign change). Proposition 3.14 implies that

$$\mathcal{L}_X \mathrm{d}x_j(\partial_k) = -\mathrm{d}x_j([X, \partial_k]).$$

Now, if we write $X = \sum_{i=1}^n a_i\partial_i$ then

$$[X, \partial_k] = -\sum_{i=1}^n \frac{\partial a_i}{\partial x_k}\partial_i$$

using the formula for the Lie bracket in Definition 3.7 so

$$-\mathrm{d}x_j([X, \partial_k]) = \frac{\partial a_j}{\partial x_k}.$$

Hence

$$\mathcal{L}_X \mathrm{d}x_j = \sum_{k=1}^n \frac{\partial a_j}{\partial x_k}\mathrm{d}x_k = \mathrm{d}(\mathrm{d}x_j(X)).$$

The Lie derivative of forms looks very difficult to compute in general, just like the Lie derivative of vector fields, but just as for vector fields we have a key result known as Cartan's formula which helps us enormously.

Proposition 4.14. *Let X be a vector field and ω a k-form on M. We define the interior product of X with ω, $i_X\omega$, to be the $(k-1)$-form defined by*

$$(i_X\omega)(p)(Y_1, \ldots, Y_{k-1}) = \omega(p)(X(p), Y_1, \ldots, Y_{k-1})$$

for all $p \in M$ and $Y_1, \ldots, Y_{k-1} \in T_pM$.

Then Cartan's formula is

$$\mathcal{L}_X\omega = \mathrm{d}(i_X\omega) + i_X(\mathrm{d}\omega).$$

Just as for Proposition 3.14 we omit the proof because the easiest way is simply a messy calculation. As a hint, notice that we have proven Cartan's formula for functions and 1-forms on $\mathbb{R}^n$ in the examples above so, together with local coordinate calculations and induction, one can quite easily now prove Cartan's formula.

Cartan's formula turns out to be surprisingly useful and worth committing to memory. Let us see how it helps in practice.

Example. Consider the 1-forms

$$\xi = \frac{x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1}{x_1^2 + x_2^2} \quad \text{and} \quad \eta = x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1$$

and vector fields

$$X = x_2 \partial_1 - x_1 \partial_2 \quad \text{and} \quad Y = x_1 \partial_1 + x_2 \partial_2$$

on $\mathbb{R}^2 \setminus \{(0,0)\}$. The vector field X corresponds to rotation and Y corresponds to dilation in $\mathbb{R}^2$.

We see that

$$\mathrm{d}\xi = 0 \quad \text{and} \quad \mathrm{d}\eta = 2\mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

By Cartan's formula:

$$\mathcal{L}_X \xi = \mathrm{d}(\xi(x_2\partial_1 - x_1\partial_2)) = \mathrm{d}(1) = 0$$

and

$$\mathcal{L}_Y \xi = \mathrm{d}(\xi(x_1\partial_1 + x_2\partial_2)) = \mathrm{d}(0) = 0$$

This means that ξ is invariant under X and Y. In contrast

$$\begin{aligned} \mathcal{L}_X \eta &= i_X(2\mathrm{d}x_1 \wedge \mathrm{d}x_2) + \mathrm{d}(\eta(x_2\partial_1 - x_1\partial_2)) \\ &= 2(x_1 \mathrm{d}x_1 + x_2 \mathrm{d}x_2) - \mathrm{d}(x_1^2 + x_2^2) = 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_Y \eta &= i_Y(2\mathrm{d}x_1 \wedge \mathrm{d}x_2) + \mathrm{d}(\eta(x_1\partial_1 + x_2\partial_2)) \\ &= 2x_1 \mathrm{d}x_2 - 2x_2 \mathrm{d}x_1 + \mathrm{d}(0) = 2x_1 \mathrm{d}x_2 - 2x_2 \mathrm{d}x_1. \end{aligned}$$

Thus η is invariant under X, but not Y.

5 Orientation

We now discuss the important notion of orientation on a manifold: this is not clear how to define this at first because there is no ambient space to determine the orientation in general, unlike say for surfaces in $\mathbb{R}^3$. A key tool in the discussion will be partitions of unity, which are very useful in their own right, including when talking about Riemannian metrics later.

5.1 Partitions of unity

It is sometimes helpful, as we have seen, to go from local definitions on charts to global definitions on the manifold (such as for the exterior derivative). It is not always possible to do this, and the constructions we have used so far are rather ad hoc – they need the particular properties of the object we plan to construct to show they are well-defined. However, we have a more general technique for going from local to global objects using something called a *partition of unity*. This allows us to break up any globally defined objects on M into pieces which are each defined just on a chart.

The result we need is the following.

Theorem 5.1. *Let M be a manifold with an atlas $\{(U_i, \varphi_i) : i \in I\}$. There exists a family of smooth functions $\{f_j : M \to \mathbb{R} : j \in \mathbb{N}\}$ such that*

- *$\forall j \in \mathbb{N}\ \exists i \in I$ such that $\operatorname{supp} f_j = \overline{\{p \in M : f_j(p) \neq 0\}} \subseteq U_i$ (the complement of $\operatorname{supp} f_j$ is the largest open set on which f_j vanishes);*
- *$\forall p \in M\ \exists$ an open set $W \ni p$ such that $W \cap \operatorname{supp} f_j \neq \emptyset$ for only finitely many $j \in \mathbb{N}$ (locally finite);*
- *$f_j(p) \geq 0$ for all $j \in \mathbb{N}$ for all $p \in M$;*
- *$\sum_{j=1}^{\infty} f_j(p) = 1$ for all $p \in M$. (Notice that this is not confusing because only finitely many of the f_k are non-zero on $W \ni p$ so the sum is always finite – this is where the locally finite property is crucial.)*

We call $\{f_j : j \in \mathbb{N}\}$ a partition of unity (subordinate to the atlas $\{(U_i, \varphi_i) : i \in I\}$).

(The rest of this subsection is not examinable.) To prove Theorem 5.1 needs two separate tools. The first is topological and we will just assume it since this is not a topology course and it only consists of standard topological facts. The second is analytic and we will discuss this in a little detail. The point is that we need model functions which vanish outside some closed ball but are identically 1 inside some smaller ball. We will now construct such functions.

Proposition 5.2. *Let $B_r(0), \overline{B_r(0)} \subseteq \mathbb{R}^n$ be the open and closed balls of radius $r > 0$ about 0. For each $r > 0$ there exists a smooth function $g_r : \mathbb{R}^n \to \mathbb{R}$ such that*

- *$g_r \geq 0$,*
- *$g_r = 1$ on $\overline{B_{\frac{r}{2}}(0)}$,*
- *$g_r = 0$ on $\mathbb{R}^n \setminus B_r(0)$*

so that $\operatorname{supp} g_r \subseteq \overline{B_r(0)}$.

Proof. Consider the function $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

As $h'(t) = \frac{1}{t^2} e^{-\frac{1}{t}} > 0$ for $t > 0$, we see that h is increasing and $0 \leq h < 1$.

Moreover $h'(t) \to 0$ as $t \to 0$ because for $t > 0$

$$t^{-k}e^{-\frac{1}{t}} \le t.t^{-k+1}e^{-\frac{1}{t}} \le (k+1)!t\sum_{m=0}^{\infty}\frac{t^{-m}}{m!}e^{-\frac{1}{t}} = (k+1)!t,$$

so sending $t \to 0$ gives the result by taking $k = 2$.

Clearly the jth derivative $h^{(j)}(t) = p_j(t)e^{-\frac{1}{t}}$ where $p_j(t)$ is a polynomial in t and t^{-1}, so the same argument shows that $h^{(j)}(t) \to 0$ as $t \to 0$, so h is a smooth function on $\mathbb{R}$ (which does not have a convergent power series expansion about 0, i.e. it is not real analytic).

Now consider $h_r : \mathbb{R} \to \mathbb{R}$ given by

$$h_r(t) = \frac{h(r^2 - t^2)}{h(r^2 - t^2) + h(t^2 - \frac{1}{4}r^2)}.$$

This is well-defined because if $h_r(r^2 - t^2) = 0$ then $|t| \ge r$ so $t^2 - \frac{1}{4}r^2 > 0$ and similarly for the other possibility. Since the denominator never vanishes and h is smooth we see that h_r is smooth as well.

Moreover, $0 \le h_r \le 1$ since $h_r \ge 0$, $h_r(t) = 0$ if and only if $|t| \ge r$ and $h_r(t) = 1$ if and only if $h(t^2 - \frac{1}{4}r^2) = 0$, which is when $|t| \le \frac{r}{2}$.

We can now define $g_r : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = h_r(|x|)$. □

The functions g_r are often called "bump functions".

We can now outline how to prove Theorem 5.1.

Proof. **(Not examinable).** Let M be an n-dimensional manifold and let $\{(U_i, \varphi_i) : i \in I\}$ be an atlas. Since U_i is homeomorphic to an open set $\varphi_i(U_i)$ in $\mathbb{R}^n$, we see that M is locally compact, i.e. for every $p \in M$ there is an open set U and compact set K such that $p \in U \subseteq K$.

It is not hard to show that our Hausdorff, second countable, locally compact space M is paracompact, i.e. any cover using open sets has a locally finite open refinement. Using the second countable property of M means that we can in fact choose a countable locally finite open refinement $\{(V_j, \psi_j) : j \in \mathbb{N}\}$ such that $\psi_j(V_j)$ is an open ball in $\mathbb{R}^n$. Moreover, by rescaling and translating we can assume that $\psi_j(V_j) = B_3(0)$ for all $j \in \mathbb{N}$ and with a bit more work we can also arrange that if $W_j = \psi_j^{-1}(B_1(0))$ then $\cup_{j=1}^{\infty} W_j = M$. Doing all this requires the fact that we can write M as a countable (nested) union of open sets which have compact closure. For example, we can write $\mathbb{R}^n = \cup_{i=1}^{\infty} B_i(0)$.

On $\psi_j(V_j)$ we have the function g_2 given by Proposition 5.2 which vanishes on $B_3(0) \setminus \overline{B_2(0)}$ and is equal to 1 on $B_1(0)$. We can therefore define a smooth function $h_j : M \to \mathbb{R}$ by

$$h_j(p) = \begin{cases} g_2(\psi_j(p)) & p \in V_j \\ 0 & p \notin V_j \end{cases}$$

Clearly $\operatorname{supp} h_j \subseteq V_j$, $0 \le h_j \le 1$ and h_j is 1 on W_j.

Now observe that h given by

$$h(p) = \sum_{j=1}^{\infty} h_j(p)$$

is a well-defined smooth function on M because for each $p \in M$ there exists an open set $W \ni p$ such that $W \cap V_j \ne \emptyset$ for only finitely many $j \in \mathbb{N}$, so $\sum_{j=1}^{\infty} h_j(p)$ is always a finite sum for each $p \in M$. Moreover, $h(p) > 0$ for all $p \in M$ because $\cup_{j=1}^{\infty} W_j = M$ so for all $p \in M$ there exists some $j \in \mathbb{N}$ such that $p \in W_j$ and thus $h_j(p) = 1$.

We finally define a partition of unity $f_j : M \to \mathbb{R}$ by $f_j = h_j/h$. □

Remark. (Not examinable). The topological conditions we have chosen to define manifolds are not far off from being necessary and sufficient for the existence of partitions of unity, which is another key reason for the choice of definition of manifolds.

5.2 Orientability and volume forms

We now want to discuss orientation on a manifold, which is an important *global* feature of a manifold.

Definition 5.3. A manifold M is *orientable* if there exists an atlas $\{(U_i, \varphi_i) : i \in I\}$ such that whenever $U_i \cap U_j \neq \emptyset$ the Jacobian

$$\det((\varphi_j \circ \varphi_i^{-1})_*) > 0$$

on $\varphi_i(U_i \cap U_j)$. (Here $\varphi_j \circ \varphi_i^{-1}$ is a map between two open subsets of $\mathbb{R}^n$ and thus a change of variables, so $(\varphi_j \circ \varphi_i^{-1})_*$ can be written as an $n \times n$ matrix and we can take the determinant of this matrix, which is the Jacobian of the change of variables.)

An *orientation* on M is a choice of such an atlas.

Example. $\mathbb{R}^n$ is orientable because we can choose an atlas with one chart (U, φ) (which is $(\mathbb{R}^n, \mathrm{id})$) and then $\det((\varphi \circ \varphi^{-1})_*) = \det \mathrm{id} = 1 > 0$.

This example shows that any manifold defined by a single chart is orientable.

Example. The n-sphere $\mathcal{S}^n$ is orientable. Let us just look at the case of $\mathcal{S}^2$ as $\mathcal{S}^n$ is the same. Take the atlas $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ we constructed, then the transition map is $F = \varphi_N \circ \varphi_S^{-1} : y \mapsto \frac{y}{|y|^2}$ on $\mathbb{R}^2 \setminus \{0\}$ so

$$F_* = \frac{1}{|y|^4} \begin{pmatrix} y_2^2 - y_1^2 & -2y_1y_2 \\ -2y_1y_2 & y_1^2 - y_2^2 \end{pmatrix}.$$

We quickly compute

$$\det F_* = \frac{1}{|y|^8}(-(y_1^2 - y_2^2)^2 - 4y_1y_2) = -\frac{1}{|y|^4} < 0.$$

To make it everywhere positive, we can switch the sign of one of the coordinates, say y_1, in the definition of φ_N, so $\varphi_N(x_1, x_2, x_3) = \frac{(-x_1, x_2)}{1-x_3}$, and then F becomes $(y_1, y_2) \mapsto \frac{(-y_1, y_2)}{|y|^2}$, so we change the sign of the first row in F_* and thus the sign of the determinant.

Of course, the argument in the example above shows that any manifold which can be covered by two charts so that the intersection is connected is always orientable, but we may have to change the definition of one of the chart maps to get the Jacobian of the transition map to be positive. This indicates the global nature of the problem of orientation: given any pair of charts we can ensure the positivity of the Jacobian of the transition map, but we cannot necessarily make a consistent choice for every possible transition map.

Example. T^n is orientable. One can explicitly write the charts, but we will see a better argument.

Example. The Möbius band and the Klein bottle are not orientable, as we shall see.

Example. All Lie groups are orientable.

Clearly, showing that a manifold is orientable using charts is a pain, so we need an easier way. We now have the following important fact which uses partitions of unity.

Theorem 5.4. *For an n-dimensional manifold M the following are equivalent:*

(a) M is orientable;

(b) there exists a nowhere vanishing n-form on M (which is called a volume form);

(c) $\Lambda^n T^*M$ is trivial.

Proof. (b)$\Leftrightarrow$(c). This is a consequence of Proposition 3.5 (a bundle of rank m is trivial if and only if it has m linearly independent sections) because $\Lambda^n T^*M$ is a rank 1 vector bundle.

(b)⇒(a). Suppose there exists a nowhere vanishing n-form Ω on M and let $\{(U_i, \varphi_i) : i \in I\}$ be an atlas where $\varphi_i(U_i)$ is connected (i.e. it does not split into two disjoint open sets), which we can always ensure. Then if we let

$$\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$$

we can compare this to the form on $\mathbb{R}^n$ we get by pullback of $\Omega|_{U_i}$ and see that

$$(\varphi_i^{-1})^*(\Omega|_{U_i}) = \lambda_i \Omega_0$$

for some nowhere vanishing function λ_i on $\varphi_i(U_i)$. Since $\varphi_i(U_i)$ is connected, λ_i is either always positive or always negative. If it is negative, we can change $\varphi_i : p \mapsto (x_1(p), \ldots, x_n(p))$ to $\varphi_i : p \mapsto (-x_1(p), x_2(p), \ldots, x_n(p))$, so that λ_i changes to $-\lambda_i$.

Now, for a transition function we have that

$$(\varphi_j \circ \varphi_i^{-1})^* \Omega_0 = \det((\varphi_j \circ \varphi_i^{-1})_*)\Omega_0$$

by one of our earlier examples; that is, the factor which comes out in the change of variables is the Jacobian. Moreover,

$$(\varphi_j \circ \varphi_i^{-1})^* \circ (\varphi_j^{-1})^*(\Omega) = (\varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1})^*\Omega = (\varphi_i^{-1})^*\Omega.$$

The left-hand side is equal to

$$\lambda_j(\varphi_j \circ \varphi_i^{-1})^*\Omega_0 = \lambda_j \det((\varphi_j \circ \varphi_i^{-1})_*)\Omega_0$$

and the right-hand side is $\lambda_i \Omega_0$, so comparing both sides we get:

$$\lambda_j \det((\varphi_j \circ \varphi_i^{-1})_*) = \lambda_i$$

on $\varphi_i(U_i \cap U_j)$. Since the Jacobian is the ratio of two positive functions (namely λ_i and λ_j) it is positive, hence $\{(U_i, \varphi_i) : i \in I\}$ is an orientation and thus M is orientable.

(a)⇒(b). Going the other way needs partitions of unity. Suppose that M is orientable and let $\{(U_i, \varphi_i) : i \in I\}$ be an orientation. By Theorem 5.1 we have a partition of unity $\{f_j : M \to \mathbb{R} : j \in \mathbb{N}\}$ which is subordinate to the atlas. For each $j \in \mathbb{N}$ choose $U_{i(j)}$ such that $\operatorname{supp} f_j \subseteq U_{i(j)}$. Since $\sum_{j=1}^{\infty} f_j = 1$, for all $p \in M$ there exists $j \in \mathbb{N}$ such that $f_j(p) \neq 0$ and hence $p \in U_{i(j)}$. Hence $\cup_{j=1}^{\infty} U_{i(j)} = M$ and so $\{(U_{i(j)}, \varphi_{i(j)}) : j \in \mathbb{N}\}$ is an orientation.

Define

$$\Omega = \sum_{j=1}^{\infty} f_j \varphi_{i(j)}^* \Omega_0,$$

where we set $f_j \varphi_{i(j)}^* \Omega_0 = 0$ outside $\operatorname{supp} f_j$ (when $f_j = 0$). This is a well-defined n-form because near each point (i.e. in some open set containing the point) only finitely many of the f_j are non-zero so the sum is always finite, but we need to show that it is nowhere vanishing.

Let $p \in M$ and let $W \ni p$ be such that $W \cap \operatorname{supp} f_j \neq \emptyset$ for only finitely many j. By taking the intersection with a coordinate chart if necessary we can suppose there exists $k \in I$ such that $W \subseteq U_k$. Then

$$(\varphi_k^{-1})^*(\Omega) = \sum_{j=1}^{\infty} f_j (\varphi_k^{-1})^* \circ \varphi_{i(j)}^* \Omega_0 = \sum_{j=1}^{\infty} f_j (\varphi_{i(j)} \circ \varphi_k^{-1})^* \Omega_0$$

and the sum is actually finite. Since the Jacobian of the transition function is positive by assumption, each of the forms

$$(\varphi_{i(j)} \circ \varphi_k^{-1})^* \Omega_0 = \det((\varphi_{i(j)} \circ \varphi_k^{-1})_*)\Omega_0$$

is a positive multiple of Ω_0. Since $f_j(p) \geq 0$ and is positive for at least one j, $(\varphi_k^{-1})^*(\Omega)(\varphi_k(p)) \neq 0$ which means that Ω does not vanish at p. □

Example. Any parallelizable n-dimensional manifold is orientable. Since TM is trivial, we have that T^*M is trivial and so we have n linearly independent 1-forms $\omega_1, \ldots, \omega_n$. Let $\Omega = \omega_1 \wedge \ldots \wedge \omega_n$. This is a volume form on M so M is orientable.

In particular, T^n is orientable.

Since Lie groups are parallelizable, all Lie groups are orientable.

Example. The restriction to $\mathcal{S}^1 \subseteq \mathbb{R}^2$ of the 1-form

$$x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1$$

on $\mathbb{R}^2$ to $\mathcal{S}^1$ is a volume form on $\mathcal{S}^1$, which we called $\mathrm{d}\theta$ before.

Similarly, then the restriction of the 2-form

$$x_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + x_2 \mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2$$

on $\mathbb{R}^3$ to $\mathcal{S}^2$ is a volume form.

We can obviously make a similar definition in higher dimensions to obtain a volume form on $\mathcal{S}^n$.

Example. On $\mathbb{R}^n$ we have the standard orientation given by $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. We can then say that an ordered basis $\{X_1, \ldots, X_n\}$ for $\mathbb{R}^n$ is *positively oriented* if $\Omega_0(X_1, \ldots, X_n) > 0$, so that the standard basis given by $X_i = \partial_i$ is positively oriented. We can write any X_i as $X_i = \sum_{j=1}^n a_{ij}\partial_j$, so the basis given by the X_i is positively oriented if and only if $\det(a_{ij}) > 0$.

Given an oriented manifold M, so it has a volume form Ω, we can define an ordered basis (or *frame*) $\{X_1, \ldots, X_n\}$ of T_pM to be *positively oriented* if $\Omega(X_1, \ldots, X_n) > 0$. In this way, we can define an orientation on each tangent space T_pM (in the usual sense) which varies smoothly over the manifold M.

Definition 5.5. We say that two orientations on M given by Ω and Ω' are the same if $\Omega' = \lambda\Omega$ for some smooth positive function $\lambda : M \to \mathbb{R}$. We see that this means that Ω' and Ω define the same orientations on every tangent space.

We say that a diffeomorphism $f : M \to N$ between oriented manifolds is *orientation preserving* if, given volume forms Ω on M and Υ on N,

$$f^*\Upsilon = \lambda\Omega$$

for some smooth positive function λ. So, $f^*\Upsilon$ is a volume form which gives the same orientation as Ω.

Notice that f is a diffeomorphism so

$$f^*\Upsilon(p)(X_1, \ldots, X_n) = \Upsilon\big(\mathrm{d}f_p(X_1), \ldots, \mathrm{d}f_p(X_n)\big)$$

is non-vanishing as $\mathrm{d}f_p$ is an isomorphism. Moreover, if we choose positively oriented bases $\{X_1, \ldots, X_n\}$ for T_pM and $\{Y_1, \ldots, Y_n\}$ for $T_{f(p)}N$ for each $p \in M$, then we see that

$$f^*\Upsilon(p)(X_1, \ldots, X_n) = \det(\mathrm{d}f_p)\Upsilon(f(p))(Y_1, \ldots, Y_n)$$

so that f is orientation preserving if and only if $\det(\mathrm{d}f_p) > 0$ (where we think of $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ with the given orientations and bases which allows us to view the differential as a matrix).

Example. The identity $\mathrm{id} : M \to M$ is always orientation preserving, trivially.

However, if we look at $-\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ we see that $(-\mathrm{id})^*\Omega_0 = \det(-\mathrm{id})\Omega_0 = (-1)^n \mathrm{id}$. Therefore $-\mathrm{id}$ is orientation preserving on $\mathbb{R}^n$ if and only if n is even (and *orientation reversing* otherwise).

This fact shows that $-\mathrm{id} : \mathcal{S}^n \to \mathcal{S}^n$ is orientation preserving if and only if n is odd, since if Ω is a volume form on $\mathcal{S}^n$ we have that $(-\mathrm{id})^*\Omega = (-1)^{n+1}\Omega$ as $\mathcal{S}^n \subseteq \mathbb{R}^{n+1}$.

Suppose that $\mathbb{RP}^n$ is orientable. Then there is a volume form Ω on $\mathbb{RP}^n$. The projection map $\pi : \mathcal{S}^n \to \mathbb{RP}^n$ is a local diffeomorphism so $\Upsilon = \pi^*\Omega$ is an n-form on $\mathcal{S}^n$. If $X_1, \ldots, X_n$ is a basis for

$T_p\mathcal{S}^n$ then $\mathrm{d}\pi_p(X_1),\ldots,\mathrm{d}\pi_p(X_n)$ is a basis for $T_{\pi(p)}\mathbb{RP}^n$ as $\mathrm{d}\pi_p : T_p\mathcal{S}^n \to T_{\pi(p)}\mathbb{RP}^n$ is an isomorphism. Hence,

$$\Upsilon(p)(X_1,\ldots,X_n) = \pi^*\Omega(p)(X_1,\ldots,X_n) = \Omega(\pi(p))(\mathrm{d}\pi_p(X_1),\ldots,\mathrm{d}\pi_p(X_n)) \neq 0$$

and so Υ is a volume form on $\mathcal{S}^n$. However, $\pi\circ(-\mathrm{id}) = \pi$ so $\pi^* = (-\mathrm{id})^*\circ\pi^*$, which means

$$\Upsilon = \pi^*\Omega = (-\mathrm{id})^*\pi^*\Omega = (-\mathrm{id})^*\Upsilon = (-1)^{n+1}\Upsilon.$$

This is a contradiction if n is even, so we must have that $\mathbb{RP}^n$ is not orientable if n is even.

Suppose that n is odd and let Υ be a volume form on $\mathcal{S}^n$, which exists as $\mathcal{S}^n$ is orientable. Then we can define Ω on $\mathbb{RP}^n$ by

$$\Upsilon(p)(\mathrm{d}\pi_p^{-1}(Y_1),\ldots,\mathrm{d}\pi_p^{-1}(Y_n)) = \Omega(\pi(p))(Y_1,\ldots,Y_n).$$

In other words, we have defined Ω so that $\pi^*\Omega = \Upsilon$. This is well-defined because if we choose $-p$ instead of p on the left-hand side we get

$$\begin{aligned}\Upsilon(-p)(\mathrm{d}\pi_{-p}^{-1}(Y_1),\ldots,\mathrm{d}\pi_{-p}^{-1}(Y_n)) &= \Upsilon(-\mathrm{id}(p))(\mathrm{d}(-\mathrm{id})_{-p}\circ\mathrm{d}\pi_p^{-1}(Y_1),\ldots,\mathrm{d}(-\mathrm{id})_{-p}\mathrm{d}\pi_p^{-1}(Y_n))\\ &= (-\mathrm{id})^*\Upsilon(p)(\mathrm{d}\pi_p^{-1}(Y_1),\ldots,\mathrm{d}\pi_p^{-1}(Y_n))\\ &= (-1)^{n+1}\Upsilon(p)(\mathrm{d}\pi_p^{-1}(Y_1),\ldots,\mathrm{d}\pi_p^{-1}(Y_n))\\ &= \Upsilon(p)(\mathrm{d}\pi_p^{-1}(Y_1),\ldots,\mathrm{d}\pi_p^{-1}(Y_n))\\ &= \Omega(\pi(p))(Y_1,\ldots,Y_n)\end{aligned}$$

since n is odd. Therefore, $\mathbb{RP}^n$ is orientable for n odd.

Example. Since the cylinder $C\subseteq\mathbb{R}^3$ and the torus $T^2\subseteq\mathbb{R}^3$ are orientable, the same argument as for $\mathbb{RP}^n$ for n even shows that the Möbius band $M = C/\mathbb{Z}_2$ and the Klein bottle $K = T^2/\mathbb{Z}_2$ are not orientable.

It is important to note that for any non-orientable n-dimensional manifold one can always a "smallest" orientable n-dimensional manifold associated to it in the following sense.

Proposition 5.6. *Let M be an n-dimensional manifold. There exists an n-dimensional manifold $\overline{M}$ such that:*

- *$\overline{M}$ is orientable;*
- *there is a covering map $\pi : \overline{M}\to M$ such that $\pi^{-1}(p)$ consists of two points for all $p\in M$;*
- *$\overline{M}$ is connected if and only if M is non-orientable.*

The manifold $\overline{M}$ is called the orientable double cover of M.

Remark. Notice that if M is orientable then the orientable double cover $\overline{M}$ is diffeomorphic to the disjoint union of two copies of M.

Example. For $\mathbb{RP}^n$ the orientable double cover $\overline{\mathbb{RP}^n}$ is diffeomorphic to $\mathcal{S}^n$ if n is even and $\mathbb{RP}^n\sqcup\mathbb{RP}^n$ if n is odd.

6 Integration

In this section we want to discuss how to integrate forms on manifolds. This will lead us to the notion of manifolds with boundary and one of the main theorems of the course: Stokes Theorem. The classical integration theorems we know about from multivariable calculus, such as the Divergence Theorem, turn out to be special cases of Stokes Theorem.

6.1 Integration on manifolds

We want to integrate n-forms ω over n-dimensional manifolds M and to define it we want to perform the now familiar trick of using usual integration on $\mathbb{R}^n$. To avoid technical analytic issues involving integration over unbounded domains we will restrict ourselves to compactly supported forms. Much of what we say can obviously be extended beyond the compact support setting, with suitable additional hypotheses about integrability of the forms.

Suppose first that ω has compact support contained in some chart (U, φ). Then

$$(\varphi^{-1})^*\omega = a\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n,$$

for some smooth function a with compact support in $\varphi(U) \subseteq \mathbb{R}^n$ (since the space of n-forms on $\mathbb{R}^n$ is spanned by $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$). We can then define

$$\begin{aligned}\int_M \omega = \int_U \omega &= \int_{\varphi(U)} (\varphi^{-1})^*\omega \\ &= \int_{\varphi(U)} a\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \\ &= \int \ldots \int_{\varphi(U)} a\mathrm{d}x_1 \ldots \mathrm{d}x_n,\end{aligned}$$

where the last line denotes the usual repeated integral on $\mathbb{R}^n$: here is where we see one of the main reasons why we use the notation $\mathrm{d}x_i$ for 1-forms. We of course have to check that we get the same answer if we choose a different chart, so suppose we have (V, ψ) so that $U \cap V \neq \emptyset$. Then, if we write

$$(\psi^{-1})^*\omega = b\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n,$$

for some function b, then we know that, on the one hand, on $\varphi(U \cap V)$ we have

$$\begin{aligned}a\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n &= (\varphi^{-1})^*\omega \\ &= (\psi \circ \varphi^{-1})^* \circ (\psi^{-1})^*\omega \\ &= (\psi \circ \varphi^{-1})^* \left(b\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n\right) \\ &= (b \circ (\psi \circ \varphi^{-1})) \det((\psi \circ \varphi^{-1})_*)\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n,\end{aligned}$$

but on the other hand

$$\begin{aligned}\int_{\psi(U\cap V)} (\psi^{-1})^*\omega &= \int \ldots \int_{\psi(U\cap V)} b\mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &= \int \ldots \int_{(\psi\circ\varphi^{-1})\circ\varphi(U\cap V)} b\mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &= \int \ldots \int_{\varphi(U\cap V)} b \circ (\psi \circ \varphi^{-1})|\det((\psi \circ \varphi^{-1})_*)|\mathrm{d}x_1 \ldots \mathrm{d}x_n,\end{aligned}$$

by the change of variables formula. Therefore, we see that the integral $\int_U \omega$ will be well-defined on overlapping charts if $\det((\psi \circ \varphi^{-1})_*) > 0$, which is exactly the condition we required for orientations. This enables us to define integration over oriented manifolds.

Definition 6.1. Let M be an oriented n-dimensional manifold with orientation given by $\{(U_i,\varphi_i):i\in I\}$, and let $\{f_j:j\in\mathbb{N}\}$ be a partition of unity subordinate to the orientation. Given a compactly supported n-form ω on M we define

$$\int_M\omega=\sum_{j=1}^{\infty}\int_M f_j\omega.$$

This definition depends on the orientation but is independent of the choice of partition of unity. If $\{g_j:j\in\mathbb{N}\}$ is another partition of unity subordinate to the orientation then

$$\begin{aligned}\sum_{j=1}^{\infty}\int_M g_j\omega&=\sum_{j=1}^{\infty}\int_M\sum_{k=1}^{\infty}g_jf_k\omega\\&=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_M g_jf_k\omega\\&=\sum_{k=1}^{\infty}\int_M\sum_{j=1}^{\infty}g_jf_k\omega\\&=\sum_{k=1}^{\infty}\int_M f_k\omega.\end{aligned}$$

Here we have used that $\sum_{j=1}^{\infty}g_j=\sum_{k=1}^{\infty}f_k=1$, that g_j, f_k and thus g_jf_k has support contained in a coordinate chart for all j,k, and that the sums are locally finite.

Example. If we consider M to be the interval (a,b) in $\mathbb{R}$ and $\Omega_0=\mathrm{d}x_1$, then obviously

$$\int_M\Omega_0=\int_a^b\mathrm{d}x_1=b-a,$$

which is just the length of the interval M.

If we take M to be a bounded open set in $\mathbb{R}^2$ and $\Omega_0=\mathrm{d}x_1\wedge\mathrm{d}x_2$ then $\int_M\Omega_0$ is the area of M. Similarly, taking M to be a bounded open set in $\mathbb{R}^3$ and $\Omega_0=\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3$ gives that $\int_M\Omega_0$ is the volume of M.

This enables us to make sense of $\int_M\Omega_0$ for bounded open sets M in $\mathbb{R}^n$ as the n-dimensional volume of M.

Example. Given any oriented n-dimensional manifold M with orientation given by charts (U_i,φ_i), we saw by the proof of Theorem 5.4 that there is a volume form Ω on M so

$$(\varphi_i^{-1})^*\Omega=\lambda_i\Omega_0$$

for some positive functions λ_i (where $\Omega_0=\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n$ on $\mathbb{R}^n$ as usual).

Therefore, if M is compact and $\{f_j:j\in\mathbb{N}\}$ is a partition of unity subordinate to the orientation and U_j is such that $\mathrm{supp}f_j\subseteq U_j$, then

$$\begin{aligned}\int_M\Omega&=\sum_{j=1}^{\infty}\int_M f_j\Omega\\&=\sum_{j=1}^{\infty}\int_{\varphi_j(U_j)}f_j(\varphi_j^{-1})^*\Omega\\&=\sum_{j=1}^{\infty}\int_{\varphi_j(U_j)}f_j\lambda_j\mathrm{d}x_1\ldots\mathrm{d}x_n>0\end{aligned}$$

since all of the terms in the sum are non-negative and each $f_j\lambda_j$ is positive somewhere. Since any volume form on M is a nowhere vanishing multiple of Ω, it will always have a non-zero integral over connected components of M.

As usual, whilst we have a perfectly good definition, it is next to useless in practice! To get around this, we have the following useful fact.

Proposition 6.2. *Let $f : M \to N$ be an orientation preserving diffeomorphism between oriented n-dimensional manifolds. Then*

$$\int_N \omega = \int_M f^*\omega$$

for all compactly supported n-forms ω on N.

Proof. **(Not examinable).** Suppose, for simplicity, that ω has compact support in a chart (V, ψ) in the orientation on N. Suppose further that there is a chart (U, φ) in the orientation on M so that $f(U) = V$ and $\varphi = \psi \circ f$ (we can arrange this since f is an orientation preserving diffeomorphism). Then

$$\begin{aligned}\int_N \omega &= \int_{\psi(V)} (\psi^{-1})^*\omega \\ &= \int_{(\psi\circ f)(U)} (f^{-1} \circ \psi^{-1})^* \circ f^*\omega \\ &= \int_{\varphi(U)} (\varphi^{-1})^* \circ f^*\omega \\ &= \int_M f^*\omega.\end{aligned}$$

The general result follows by using appropriate partitions of unity. □

Remark. Proposition 6.2 is clearly false if we do not assume that f is orientation preserving. A simple example is if we take $f = -\operatorname{id}$ on $\mathbb{R}$ and $M = (-1, 1)$, then f is a diffeomorphism of M, but $f^* = -\operatorname{id}$ acting on 1-forms.

Example. If we define $f : (0, 2\pi) \to \mathcal{S}^1 \subseteq \mathbb{R}^2$ by $f(\theta) = (\cos\theta, \sin\theta)$ then f is a diffeomorphism onto its image and we see that if

$$\Omega = x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1$$

then, since $f^*\mathrm{d}x_j = \mathrm{d}(x_j \circ f)$,

$$\begin{aligned}f^*\Omega &= \cos\theta \mathrm{d}(\sin\theta) - \sin\theta \mathrm{d}(\cos\theta) \\ &= \cos^2\theta \mathrm{d}\theta + \sin^2\theta \mathrm{d}\theta \\ &= \mathrm{d}\theta.\end{aligned}$$

Since the complement of $f(0, 2\pi)$ in $\mathcal{S}^1$ has measure zero, we see that

$$\int_{\mathcal{S}^1} \Omega = \int_0^{2\pi} f^*\Omega = \int_0^{2\pi} \mathrm{d}\theta = 2\pi.$$

This reflects the fact that Ω restrict to be a volume form on $\mathcal{S}^1$ (in fact, one that gives the correct length for the unit circle).

Example. If we define $f : (0, \pi) \times (0, 2\pi) \to \mathcal{S}^2 \subseteq \mathbb{R}^3$ by $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ then f is a diffeomorphism onto its image and the complement of the image in $\mathcal{S}^2$ has measure zero. If we let

$$\Omega = x_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + x_2 \mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2$$

then one can see (and you should check!) that

$$f^*\Omega = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Therefore,

$$\int_{\mathcal{S}^2} \Omega = \int_0^{2\pi} \int_0^{\pi} \sin\theta \mathrm{d}\theta \mathrm{d}\phi = 4\pi,$$

which is the expected answer for the area of the unit sphere in $\mathbb{R}^3$.

6.2 Manifolds with boundary

So that we may state Stokes Theorem, we need to enlarge the class of manifolds slightly to *manifolds with boundary*. Again, we begin with the key examples.

Example. The simplest n-dimensional manifold with boundary is the n-dimensional closed upper half-space

$$\overline{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_n \geq 0\}.$$

Here, the boundary is just

$$\partial\overline{H}^n = \{(x_1, \ldots, x_{n-1}, 0) \,:\, (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\}.$$

Clearly, $\partial\overline{H}^n$ is an $(n-1)$-dimensional manifold, diffeomorphic to $\mathbb{R}^{n-1}$.

Example. A closed interval $M = [a, b]$ is a 1-dimensional manifold with boundary, and the boundary is $\partial M = \{a\} \cup \{b\}$, which is a (disconnected) 0-dimensional manifold.

Example. The closed unit ball in $\mathbb{R}^n$,

$$\overline{B}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, |x|^2 \leq 1\}$$

is an n-dimensional manifold with boundary. The boundary is $\partial\overline{B}^n = \mathcal{S}^{n-1}$, which is an $(n-1)$-dimensional manifold.

The definition of an n-dimensional manifold with boundary is identical to that of a usual manifold except that we replace $\mathbb{R}^n$ with $\overline{H}^n$.

Definition 6.3. An n-dimensional *manifold with boundary* is a (second countable, Hausdorff) topological space M such that there exists a family $\mathcal{A} = \{(U_i, \varphi_i) \,:\, i \in I\}$ where:

- U_i is an open set in M and $\cup_{i\in I} U_i = M$;
- $\varphi_i : U_i \to \overline{H}^n$ is a continuous bijection onto an open set $\varphi_i(U_i)$ with continuous inverse (i.e. a homeomorphism);
- whenever $U_i \cap U_j \neq \emptyset$, the *transition map* $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a smooth (infinitely differentiable) bijection with smooth inverse (i.e. a diffeomorphism).

The family $\mathcal{A}$ is called an *atlas* and the pairs (U_i, φ_i) are called (coordinate) *charts*. The *boundary* ∂M of M is given by

$$\partial M = \{p \in M \,:\, \exists (U_i, \varphi_i) \in \mathcal{A} \text{ such that } \varphi_i(p) \in \partial\overline{H}^n\}.$$

The boundary ∂M is an $(n-1)$-dimensional manifold with charts (V_i, ψ_i) given by $V_i = U_i \cap \partial M$ (when this is non-empty) and $\psi_i = \varphi_i|_{V_i}$ (where we identify $\partial\overline{H}^n$ with $\mathbb{R}^{n-1}$ in the obvious way).

Remark. (Not examinable). You may wonder what we mean by a smooth map between open subsets of $\overline{H}^n$: we take this to mean the restriction of a smooth map between two open subsets of $\mathbb{R}^n$.

Example. The closed cylinder $\overline{C} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 = 1, -1 \leq x_3 \leq 1\}$ is a 2-dimensional manifold with boundary $\partial\overline{C}$ which is two disjoint copies of $\mathcal{S}^1$.

If $\mathbb{Z}_2$ acts on $\mathbb{R}^3$ as $\pm\,\mathrm{id}$ as usual, then $\overline{C}/\mathbb{Z}_2$ is also often called the Möbius band, which is a 2-dimensional manifold with boundary, whose boundary is now just diffeomorphic to $\mathcal{S}^1$.

All of the definitions we have of tangent vectors, vector fields, differential forms, orientation and integration all carry through to manifolds with boundary with little modification. However, one thing we do want to understand is the relationship between orientations on M and orientations on ∂M.

Proposition 6.4. *Given an orientation on a manifold M with boundary ∂M, there is a natural orientation, called the induced orientation, on ∂M.*

Proof. Let $\{(U_i, \varphi_i) : i \in I\}$ be an orientation on M and let (V_i, ψ_i) be the charts on ∂M given by $V_i = U_i \cap \partial M$ (when this is non-empty) and $\psi_i = \varphi_i|_{V_i}$. We claim the resulting atlas on ∂M defines an orientation on ∂M.

Consider V_i, V_j such that $V_i \cap V_j \neq \emptyset$. For simplicity of notation let $f = \varphi_j \circ \varphi_i^{-1}$ and let $g = \psi_j \circ \psi_i^{-1}$. Notice that g is just the restriction of f to points in $\partial \overline{H}^n$, i.e.

$$f(x_1, \ldots, x_{n-1}, 0) = g(x_1, \ldots, x_{n-1}).$$

Since f is a diffeomorphism it must take points in $\partial \overline{H}^n$ to points in $\partial \overline{H}^n$. Hence, since ∂_n is inward pointing on $\partial \overline{H}^n$, $f_*(\partial_n)$ must also be inward pointing on $\partial \overline{H}^n$, i.e. it must have a positive component in the ∂_n direction. In other words, if we write $f = (f_1, \ldots, f_n)$ then

$$f_n(x_1, \ldots, x_{n-1}, 0) = 0 \quad \text{and} \quad \frac{\partial f_n}{\partial x_n} > 0.$$

Therefore, along $\partial \overline{H}^n$, we must have

$$f_* = \left(\frac{\partial f_i}{\partial x_j}\right) = \begin{pmatrix} g_* & A \\ 0 & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

for some column vector A, and so

$$\det f_* = \frac{\partial f_n}{\partial x_n} \det g_*.$$

Since $\det f_* > 0$ (as we assumed our atlas was an orientation) and $\frac{\partial f_n}{\partial x_n} > 0$, we deduce that $\det g_* > 0$, and thus the atlas on ∂M is an orientation as claimed. $\square$

We can also encode the induced orientation in terms of volume forms, as we show in a couple of examples.

Example. We have the standard volume form $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ on $\overline{H}^n$. The *outward unit normal* on $\partial \overline{H}^n$ is $-\partial_n$. Hence,

$$\Omega = i_{-\partial_n}\Omega_0 = (-1)^n \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{n-1}$$

is a volume form on $\partial \overline{H}^n$, which defines the induced orientation on $\partial \overline{H}^n$.

Example. We also have the standard volume form $\Omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ on $\overline{B}^2$. The outward unit normal on $\mathcal{S}^1$ is $x_1\partial_1 + x_2\partial_2$. Hence, the restriction to $\mathcal{S}^1$ of

$$\Omega = i_{x_1\partial_1 + x_2\partial_2}\Omega_0 = x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1$$

is a volume form on $\mathcal{S}^1$, which we see is the 1-form we identified with $\mathrm{d}\theta$ before.

This obviously generalises to higher dimensions, in particular the volume form we would obtain on $\mathcal{S}^2$ from this construction on $\overline{B}^3$ (with $\Omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$) is the restriction of

$$i_{x_1\partial_1 + x_2\partial_2 + x_3\partial_3}\Omega_0 = x_1\mathrm{d}x_2 \wedge \mathrm{d}x_3 + x_2\mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2,$$

which we saw earlier.

6.3 Stokes Theorem

We now have all of the tools to state and prove Stokes Theorem.

Theorem 6.5 (Stokes)**.** *Let M an n-dimensional oriented manifold with boundary, and let ∂M be endowed with the induced orientation. Let ω be a compactly supported $(n-1)$-form on M. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

As we will see from the proof, this theorem essentially boils down to the Fundamental Theorem of Calculus and so is, in some sense, "trivial". However, the reason why it is trivial is that we defined all of the terms in the statement correctly to make it trivial. Moreover, just because a theorem is trivial does not make it useless! In fact, we will see that Stokes Theorem has many interesting applications.

Proof. The idea of the proof is to use a partition of unity so that we need only consider the case of forms with compact support in charts, and hence reduce to just proving the result for forms with compact support in $\overline{H}^n$.

Suppose first that ω has compact support in some chart (U, φ). Then we may write

$$\begin{aligned}(\varphi^{-1})^*\omega &= a_1 \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n - a_2 \mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \ldots \wedge \mathrm{d}x_n + \ldots + (-1)^{n-1} a_n \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{n-1} \\ &= \sum_{i=1}^n (-1)^{i-1} a_i \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \wedge \mathrm{d}x_n,\end{aligned}$$

where the a_i are smooth functions with compact support in $\varphi(U) \subseteq \overline{H}^n$ and $\widehat{\mathrm{d}x_i}$ indicates that this term is omitted (and the $(-1)^{i-1}$ are put in for convenience, as we shall see). Notice, in particular, that only the last form in the sum is non-zero on $\partial\overline{H}^n$ and so on $\varphi(U) \cap \partial\overline{H}^n$,

$$(\varphi^{-1})^*\omega = (-1)^{n-1} a_n \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{n-1}.$$

We may then compute

$$\begin{aligned}(\varphi^{-1})^*\mathrm{d}\omega &= \mathrm{d}\big((\varphi^{-1})^*\omega\big) \\ &= \left(\frac{\partial a_1}{\partial x_1} + \ldots + \frac{\partial a_n}{\partial x_n}\right) \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \\ &= \sum_{i=1}^n \frac{\partial a_i}{\partial x_i} \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.\end{aligned}$$

(We see here why we bothered to put in the factors of $(-1)^{i-1}$ earlier.)

Since the a_i have compact support, we may choose R sufficiently large and an open set $V \subseteq \overline{H}^n$ so that $\operatorname{supp} a_i \subseteq V \subseteq [-R, R]^{n-1} \times [0, R]$ for all i. Thus, a_i vanishes on all of the faces of the cube, except perhaps $[-R, R]^{n-1} \times \{0\}$.

We now see that the integral

$$\begin{aligned}\int_M \mathrm{d}\omega &= \int_U \mathrm{d}\omega \\ &= \int_{\varphi(U)} (\varphi^{-1})^*\mathrm{d}\omega \\ &= \int_{[-R,R]^{n-1}\times[0,R]} \sum_{i=1}^n \frac{\partial a_i}{\partial x_i} \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.\end{aligned}$$

Using Fubini's Theorem to change the order of integration if necessary, and the Fundamental Theorem of Calculus, we see that, for $i < n$

$$\begin{aligned}\int_{[-R,R]^{n-1}\times[0,R]} \frac{\partial a_i}{\partial x_i} \mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_{[-R,R]^{n-2}\times[0,R]} & a_i(x_1, \ldots, x_{i-1}, R, x_{i+1}, \ldots, x_n) \\ & - a_i(x_1, \ldots, x_{i-1}, -R, x_{i+1}, \ldots, x_n) \mathrm{d}x_1 \ldots \widehat{\mathrm{d}x_i} \ldots \mathrm{d}x_n = 0,\end{aligned}$$

since a_i vanishes on any face of the cube where x_n is not identically 0. Similarly,

$$\begin{aligned}\int_{[-R,R]^{n-1}\times[0,R]}\frac{\partial a_n}{\partial x_n}\mathrm{d}x_1\ldots\mathrm{d}x_n &= \int_{[-R,R]^{n-1}} a_n(x_1,\ldots,x_{n-1},R)-a_n(x_1,\ldots,x_{n-1},0)\mathrm{d}x_1\ldots\mathrm{d}x_{n-1}\\ &= -\int_{[-R,R]^{n-1}} a_n(x_1,\ldots,x_{n-1},0)\mathrm{d}x_1\ldots\mathrm{d}x_{n-1}\\ &= \int_{[-R,R]^{n-1}}(-1)^{n-1}a_n\big((-1)^n\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_{n-1}\big)\\ &= \int_{\varphi(U)\cap\partial\overline{H}^n}(\varphi^{-1})^*\omega\\ &= \int_{\partial M}\omega,\end{aligned}$$

where we recall that the induced orientation on $\partial\overline{H}^n$ has volume form $(-1)^n\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_{n-1}$.

We deduce that Stokes Theorem holds for ω with compact support in a chart.

Given a general compactly supported form ω as in the statement, choose a partition of unity $\{f_j : j\in\mathbb{N}\}$ subordinate to the orientation. Then $f_j\omega$ has compact support in some chart and therefore Stokes Theorem holds for $f_j\omega$. We therefore see that

$$\begin{aligned}\int_M\mathrm{d}\omega &= \int_M\mathrm{d}\left(\sum_{j=1}^\infty f_j\omega\right)\\ &= \sum_{j=1}^\infty\int_M\mathrm{d}(f_j\omega)\\ &= \sum_{j=1}^\infty\int_{\partial M}f_j\omega\\ &= \int_{\partial M}\sum_{j=1}^\infty f_j\omega\\ &= \int_{\partial M}\omega,\end{aligned}$$

where again we use the fact that all of the sums are locally finite. The result follows. □

An important immediate corollary is the following since an ordinary manifold has empty boundary and Stokes Theorem applies (or one can just easily adapt the proof above).

Corollary 6.6. *Let M be an oriented n-dimensional manifold and let ω be a compactly supported $(n-1)$-form. Then*

$$\int_M\mathrm{d}\omega=0.$$

Example. We saw before that we had a 1-form on Ω on $\mathcal{S}^1$ so that if $f(\theta)=(\cos\theta,\sin\theta)$ then $f^*\Omega=\mathrm{d}\theta$ and that

$$\int_{\mathcal{S}^1}\Omega=2\pi.$$

Since this is non-zero we see that Ω cannot be exact, i.e. $\mathrm{d}f$ for some function f (despite the fact that $f^*\Omega=\mathrm{d}\theta$), although Ω is closed.

More generally, we see that no volume form Ω on a compact oriented manifold can be exact (since $\int_M\Omega\neq0$ for each connected component M of the manifold), although $\mathrm{d}\Omega=0$.

The significance of Corollary 6.6 and this example will become apparent in the next section on de Rham cohomology.

Remark. Using our earlier examples relating div and curl to the exterior derivative, it is now straightforward to show that the classical Green's Theorem, Divergence Theorem and Stokes Theorem for integrals in multivariable calculus are all special cases of the Stokes Theorem we have proved.

To conclude this section we give one important application of Stokes Theorem: the Brouwer Fixed Point Theorem.

Theorem 6.7 (Brouwer Fixed Point Theorem)**.** *Every smooth map* $f : \overline{B}^n \to \overline{B}^n$ *has a fixed point.*

Proof. Suppose, for a contradiction, that f has no fixed point, so $f(x) \neq x$ for all $x \in \overline{B}^n$. Then, there is a unique line from $f(x)$ to x which can be extended until it meets $\mathcal{S}^{n-1}$: let the point where it meets $\mathcal{S}^{n-1}$ be $g(x)$. Notice that if $x \in \mathcal{S}^{n-1}$ then $g(x) = x$.

We therefore have a smooth map $g : \overline{B}^n \to \mathcal{S}^{n-1}$ so $g|_{\mathcal{S}^{n-1}} = \mathrm{id}$.

Let Ω be the standard volume form on $\mathcal{S}^{n-1}$ (so that $\int_{\mathcal{S}^{n-1}} \Omega > 0$). Then $g^*\Omega$ is an $(n-1)$-form on $\overline{B}^n$ so that on $\mathcal{S}^{n-1}$ we have $g^*\Omega = \Omega$ (as g is the identity on $\mathcal{S}^{n-1}$). Moreover, since $\mathrm{d}\Omega = 0$, we have that $\mathrm{d}(g^*\Omega) = g^*(\mathrm{d}\Omega) = 0$.

By Stokes Theorem, we have

$$\begin{aligned} 0 < \int_{\mathcal{S}^{n-1}} \Omega &= \int_{\mathcal{S}^{n-1}} g^*\Omega \\ &= \int_{\partial\overline{B}^n} g^*\Omega \\ &= \int_{\overline{B}^n} \mathrm{d}(g^*\Omega) = 0, \end{aligned}$$

yielding our required contradiction. □

Remark. The Brouwer Fixed Point Theorem (and stronger versions of it) has many important applications in a variety of areas, perhaps most notably in game theory.

7 De Rham cohomology and applications

On a manifold M, we saw that we have closed k-forms

$$\mathcal{Z}^k(M) = \{\omega \in \Gamma(\Lambda^k T^*M) \,:\, \mathrm{d}\omega = 0\} = \mathrm{Ker}\big(\mathrm{d} : \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{k+1} T^*M)\big)$$

and exact k-forms

$$\mathcal{E}^k(M) = \{\mathrm{d}\eta \,:\, \eta \in \Gamma(\Lambda^{k-1} T^*M)\} = \mathrm{Im}\big(\mathrm{d} : \Gamma(\Lambda^{k-1} T^*M) \to \Gamma(\Lambda^k T^*M)\big).$$

(By definition, we take $\mathcal{E}^0(M) = 0$.) Since $\mathrm{d}^2 = 0$, we see that $\mathcal{E}^k(M) \subseteq \mathcal{Z}^k(M)$, but what's the difference between these two spaces? Thinking about this questions, perhaps surprisingly, turns out to yield a useful invariant of manifolds.

7.1 De Rham cohomology: definition and properties

Definition 7.1. We define the kth de Rham cohomology group of an n-dimensional manifold M by

$$H^k(M) = \mathcal{Z}^k(M)/\mathcal{E}^k(M),$$

i.e. the quotient of the closed k-forms by the exact k-forms. Even though this is the quotient of two infinite-dimensional vector spaces, $H^k(M)$ can be a *finite* vector space. (This will happen, for example, if M is compact but this is a highly nontrivial fact!) Notice that $H^k(M) = 0$ for $k > n$, trivially, as there are no non-zero k-forms on M for $k > n$.

We denote elements of $H^k(M)$ by $[\omega]$ for $\omega \in \mathcal{Z}^k(M)$ and note that $[\omega] = [\omega']$ if and only if $\omega - \omega' = \mathrm{d}\eta$, i.e. $\omega - \omega' \in \mathcal{E}^k(M)$. In particular, $[\omega] = 0$ if and only if $\omega = \mathrm{d}\eta \in \mathcal{E}^k(M)$.

Example. We see $f \in \mathcal{Z}^0(M)$ if and only if $\mathrm{d}f = 0$, which is if and only if f is locally constant. Since $\mathcal{E}^0(M) = 0$ we see that, if M has m connected components,

$$H^0(M) \cong \mathbb{R}^m.$$

In particular, if M is connected, $H^0(M) \cong \mathbb{R}$.

Remark. Strictly speaking, we should continue to write that de Rham cohomology groups are isomorphic to Euclidean spaces of the relevant dimension, but for ease of notation we shall from now on simply write equalities, e.g. $H^0(M) = \mathbb{R}$ if M is connected.

Example. If M is a compact orientable n-manifold, we saw that any volume form on M cannot be exact and so $H^n(M) \neq 0$. Moreover, we see that if $[\omega] = [\omega'] \in H^n(M)$ then $\omega' = \omega + \mathrm{d}\eta$ so

$$\int_M \omega' = \int_M \omega + \mathrm{d}\eta = \int_M \omega$$

by Stokes Theorem.

Example. A point $M = \{*\}$ is a connected 0-dimensional manifold. Therefore,

$$H^k(\{*\}) = \begin{cases} \mathbb{R} & k = 0, \\ 0 & k > 0. \end{cases}$$

This may seem like a silly example, but actually will turn out to be important.

We now notice that there is well-defined product we can define on de Rham cohomology.

Definition 7.2. We define the cup product $\cup : H^k(M) \times H^l(M) \to H^{k+l}(M)$ by

$$[\alpha] \cup [\beta] = [\alpha \wedge \beta]$$

We see that it is well-defined because if $\alpha' = \alpha + \mathrm{d}\eta$ and $\beta' = \beta + \mathrm{d}\zeta$ then

$$\begin{aligned}\alpha' \wedge \beta' &= (\alpha + \mathrm{d}\eta) \wedge (\beta + \mathrm{d}\zeta)\\ &= \alpha \wedge \beta + \mathrm{d}(\eta \wedge \beta + (-1)^k \alpha \wedge \zeta + \eta \wedge \mathrm{d}\zeta),\end{aligned}$$

since α, β and $\mathrm{d}\zeta$ are closed. Moreover, we see that

$$[\beta] \cup [\alpha] = (-1)^{kl}[\alpha] \cup [\beta]$$

by the properties of the wedge product.

Example. We see that $[\alpha] \cup 0 = 0$ for any $[\alpha] \in H^k(M)$.

Example. If $[\alpha] \in H^1(M)$ then $[\alpha] \cup [\alpha] = 0$ since $\alpha \wedge \alpha = 0$.

Recall that we can pullback forms using a smooth map between smooth manifolds. This gives a natural map between de Rham cohomology groups which works well with the cup product.

Proposition 7.3. *Let $f : M \to N$ be a smooth map between smooth manifolds. We have well-defined linear maps $f^* : H^k(N) \to H^k(M)$ given by*

$$f^*[\omega] = [f^*\omega]$$

which satisfy

$$f^*\big([\alpha] \cup [\beta]\big) = f^*[\alpha] \cup f^*[\beta].$$

Moreover, if $f : M \to N$ is a diffeomorphism, then $f^* : H^k(N) \to H^k(M)$ is an isomorphism.

Proof. Since $\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)$, if $\omega \in \mathcal{Z}^k(N)$ then $f^*\omega \in \mathcal{Z}^k(M)$. Thus the map is well-defined, and linearity is clear.

Since $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$, we see that

$$f^*\big([\alpha] \cup [\beta]\big) = f^*[\alpha \wedge \beta] = [f^*(\alpha \wedge \beta)] = [f^*\alpha \wedge f^*\beta] = [f^*\alpha] \cup [f^*\beta] = f^*[\alpha] \cup f^*[\beta].$$

Finally, if f, g are smooth maps between smooth manifolds then $(f \circ g)^* = g^* \circ f^*$ as maps between de Rham cohomology groups (since the same is true for maps on forms). Therefore, if f is a diffeomorphism,

$$(f^{-1})^* \circ f^* = (f \circ f^{-1})^* = \mathrm{id}^* = \mathrm{id} = f^* \circ (f^{-1})^*,$$

so f^* is invertible and hence an isomorphism. □

Example. Define $f : \mathbb{R} \to \mathcal{S}^1 \subseteq \mathbb{R}^2 \setminus \{0\}$ by $f(\theta) = (\cos\theta, \sin\theta)$. Then we saw that the restriction of the 1-form

$$\xi = \frac{x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1}{x_1^2 + x_2^2}$$

to $\mathcal{S}^1$ was closed and a volume form on $\mathcal{S}^1$. Thus $[\xi] \in H^1(\mathcal{S}^1)$ and we know that $f^*\xi = \mathrm{d}\theta$. Therefore

$$f^*[\xi] = [\mathrm{d}\theta] = 0 \in H^1(\mathbb{R})$$

(since θ is a well-defined function on $\mathbb{R}$), whereas we know that $[\xi] \neq 0 \in H^1(\mathcal{S}^1)$ as $\mathcal{S}^1$ is compact and ξ is a volume form on $\mathcal{S}^1$. Therefore, f^* is certainly not an isomorphism, even though f is a local diffeomorphism.

Proposition 7.3 shows that de Rham cohomology is a diffeomorphism invariant, and so we can be used to distinguish manifolds. However, de Rham cohomology is actually invariant in a stronger sense: it is a *homotopy invariant*. We shall prove a weaker statement here, namely that if $h : M \times [0, 1] \to N$ is a smooth map and we let $h_t(p) = h(p, t)$ then $h_t^* : H^k(N) \to H^k(M)$ is independent of t.

Theorem 7.4. *Let $h : M \times [0,1] \to N$ be a smooth map and let $h_t(p) = h(p,t)$ for $p \in M$. Then the induced maps $h_t^* : H^k(N) \to H^k(M)$ satisfy $h_1^* = h_0^*$.*

Proof. **(Not examinable).** The idea is to use the Fundamental Theorem of Calculus and show that

$$h_1^*\alpha - h_0^*\alpha = \int_0^1 \frac{\partial}{\partial t} h_t^*\alpha \,\mathrm{d}t$$

is exact if α is a closed form on N. Here, we can interpret the integral as just integrating the t-dependent coefficients in the k-form $h_t^*\alpha$ on M to give a t-independent k-form on M. Thus, $h_1^*[\alpha] - h_0^*[\alpha] = [h_1^*\alpha - h_0^*\alpha] = 0$, which is what we want to show.

Let α be a k-form on N. Then we may view $\beta = h_t^*\alpha$ and $\gamma = i_{\partial_t} h^*\alpha$ as t-dependent k and $(k-1)$-forms on M, respectively: the fact that γ naturally defines a form on M is that $i_{\partial_t}\gamma = i_{\partial_t} i_{\partial_t} h^*\alpha = 0$. Moreover, we can write

$$h^*\alpha = \beta + \mathrm{d}t \wedge \gamma.$$

By the definition of the exterior derivative, we see that we can relate the exterior derivative $\mathrm{d}_{M\times[0,1]}$ to the exterior derivative d_M on M by

$$\mathrm{d}_{M\times[0,1]} = \mathrm{d}_M + \mathrm{d}t \wedge \frac{\partial}{\partial t}.$$

(One can see this easily in local coordinates.) Therefore, if $\alpha \in \mathcal{Z}^k(N)$ we see that

$$0 = h^*(\mathrm{d}\alpha) = \mathrm{d}_{M\times[0,1]} h^*\alpha = \mathrm{d}_M\beta + \mathrm{d}t \wedge \frac{\partial \beta}{\partial t} - \mathrm{d}t \wedge \mathrm{d}_M\gamma.$$

(There is no $\frac{\partial\gamma}{\partial t}$ term as $\mathrm{d}t \wedge \mathrm{d}t = 0$.) We deduce that $\beta = h_t^*\alpha \in \mathcal{Z}^k(M)$ (which we already knew, since $\alpha \in \mathcal{Z}^k(N)$) and

$$\frac{\partial \beta}{\partial t} = \mathrm{d}_M\gamma.$$

Since $\beta = h_t^*\alpha$, we can now calculate as we suggested at the beginning of the proof:

$$\begin{aligned} h_1^*\alpha - h_0^*\alpha &= \int_0^1 \frac{\partial}{\partial t} h_t^*\alpha \,\mathrm{d}t \\ &= \int_0^1 \frac{\partial \beta}{\partial t}\, dt \\ &= \int_0^1 \mathrm{d}_M\gamma \,\mathrm{d}t \\ &= \mathrm{d}_M\Big(\int_0^1 \gamma \,\mathrm{d}t\Big) \in \mathcal{E}^k(M). \end{aligned}$$

Hence $h_1^*[\alpha] - h_0^*[\alpha] = [h_1^*\alpha - h_0^*\alpha] = 0$, as desired. □

7.2 Examples

Theorem 7.4 yields the following immediately corollary, which gives us our first non-trivial calculation of de Rham cohomology.

Corollary 7.5. *The de Rham cohomology groups of $\mathbb{R}^n$ are:*

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0, \\ 0 & k > 0. \end{cases}$$

In other words, all closed k-forms on $\mathbb{R}^n$ for $k > 0$ are exact.

This says that the de Rham cohomology of $\mathbb{R}^n$ is the same as that of a point.

Proof. Since $\mathbb{R}^n$ is connected, $H^0(\mathbb{R}^n) = \mathbb{R}$.

We now consider $k > 0$ and define $h : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ by $h(x,t) = tx$. This is a smooth map so that $h_1 = \mathrm{id}$ and h_0 is the zero map. Theorem 7.4 states that $h_1^* = \mathrm{id}$ equals h_0^* on $H^k(\mathbb{R}^n)$. However, $h_0^*\omega = 0$ for any k-form ω (as $k > 0$), so $h_0^* = h_1^* = \mathrm{id}$ is the zero map on $H^k(\mathbb{R}^n)$. Therefore, $H^k(\mathbb{R}^n) = 0$ for $k > 0$. □

Remark. Corollary 7.5 gives the same answer (by the same proof) for any star-shaped region U in $\mathbb{R}^n$: that is, an open set U containing a point p (usually the origin) so that for all $x \in U$ the straight line from p to x is contained in U. This result is known as the *Poincaré Lemma*.

Example. Let M be any manifold and consider $M \times \mathbb{R}^m$. If we define $h : M \times \mathbb{R}^m \times [0,1] \to M \times \mathbb{R}^m$ by $h(p,x,t) = (p,tx)$, then $h_1 = \mathrm{id}$ and h_0 is essentially the projection map from $M \times \mathbb{R}^m$ to M. Therefore, Theorem 7.4 (as in the proof of Corollary 7.5) gives us that

$$H^k(M \times \mathbb{R}^m) \cong H^k(M)$$

for all k.

We saw that for compact, orientable, n-dimensional manifolds M that $H^n(M) \neq 0$. We can actually say something stronger.

Theorem 7.6. *Let M be a compact, connected, orientable, n-dimensional manifold. Then*

$$H^n(M) \cong \mathbb{R}.$$

Moreover, given an orientation on M, we can define this isomorphism by

$$[\omega] \mapsto \int_M \omega$$

for n-forms ω.

Remark. Theorem 7.6 is a special case of *de Rham's Theorem.*

We will postpone the proof of Theorem 7.6 as it is long and involved, just so that we can see how to use it to compute de Rham cohomology in some important cases.

Example. The de Rham cohomology groups of $\mathcal{S}^1$ are:

$$H^0(\mathcal{S}^1) = \mathbb{R} \quad \text{and} \quad H^1(\mathcal{S}^1) = \mathbb{R}.$$

This is immediate from Theorem 7.6 and the fact that $\mathcal{S}^1$ is connected.

We can now compute the de Rham cohomology groups of spheres.

Theorem 7.7. *The de Rham cohomology groups of $\mathcal{S}^n$ are:*

$$H^k(\mathcal{S}^n) = \begin{cases} \mathbb{R} & k = 0, n, \\ 0 & \textit{otherwise.} \end{cases}$$

The proof is a little long but the ideas involved are quite straightforward. It is important to understand the steps, as the techniques used are of good general use in geometry.

Proof. Recall the atlas $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ for $\mathcal{S}^n$ we introduced near the start of the course. In particular, remember that $\varphi_N(U_N) = \varphi_S(U_S) = \mathbb{R}^n$ and that $U_N \cap U_S$ is diffeomorphic to $\mathcal{S}^{n-1} \times \mathbb{R}$, which is connected if $n > 1$.

We observe that the result is true for $n = 1$, so suppose that $n > 1$. We also know that $H^0(\mathcal{S}^n) = \mathbb{R} = H^n(\mathcal{S}^n)$ by Theorem 7.6, so we just need to show that $H^k(\mathcal{S}^n) = 0$ for $0 < k < n$.

We first consider $[\alpha] \in H^1(\mathcal{S}^n)$. Then $(\varphi_N^{-1})^*\alpha$ is a closed 1-form on $\mathbb{R}^n$ and so is exact by Corollary 7.5. Hence, α is exact on U_N and similarly on U_S, so there exist smooth functions u_N and u_S on U_N and U_S so that $\alpha = \mathrm{d}u_N$ on U_N and $\alpha = \mathrm{d}u_S$ on U_S. We deduce that on $U_N \cap U_S$ we have

$$\mathrm{d}(u_N - u_S) = 0.$$

Since $U_N \cap U_S$ is connected, we have that

$$u_N - u_S = c \in \mathbb{R}$$

on $U_N \cap U_S$. Therefore, we may define a smooth function u on $\mathcal{S}^n$ by

$$u(p) = \begin{cases} u_N(p) & p \in U_N, \\ u_S(p) + c & p \in U_S. \end{cases}$$

It is well-defined because $u_N = u_S + c$ on $U_N \cap U_S$. Then, since $\mathrm{d}u = \mathrm{d}u_N = \alpha$ on U_N and $\mathrm{d}u = \mathrm{d}(u_S + c) = \mathrm{d}u_S = \alpha$ on U_S, we deduce that $\alpha = \mathrm{d}u$ and thus $[\alpha] = 0$. Therefore, $H^1(\mathcal{S}^n) = 0$. In particular, we have proved the claimed result for $n = 2$.

We therefore need only consider $1 < k < n$ and $n > 2$ from now on. We proceed by induction on n and suppose that $H^l(\mathcal{S}^{n-1}) = 0$ for $1 < l < n-1$ (and note that we also know that $H^1(\mathcal{S}^{n-1}) = 0$ by the argument above since $n-1 > 1$). Let $[\alpha] \in H^k(\mathcal{S}^n)$ for some $1 < k < n$. Then by Corollary 7.5 again, we know that there exist $(k-1)$-forms β_N and β_S on U_N and U_S so that $\alpha = \mathrm{d}\beta_N$ on U_N and $\alpha = \mathrm{d}\beta_S$ on U_S. Therefore, as before, we see that

$$\mathrm{d}(\beta_N - \beta_S) = 0$$

on $U_N \cap U_S$. Hence,

$$[\beta_N - \beta_S] \in H^{k-1}(U_N \cap U_S) = H^{k-1}(\mathcal{S}^{n-1} \times \mathbb{R}) = H^{k-1}(\mathcal{S}^{n-1}) = 0$$

by the inductive hypothesis (as $0 < k-1 < n-1$). We deduce that there exists a $(k-2)$-form γ on $U_N \cap U_S$ so that

$$\beta_N - \beta_S = \mathrm{d}\gamma$$

on $U_N \cap U_S$.

Now, we have to work slightly harder than we did above because we cannot just extend γ to $\mathcal{S}^n$ (whereas last time the analogue of γ was the constant c which we could obviously extend to $\mathcal{S}^n$). To get around this, choose a bump function f on $\mathcal{S}^n$ so that $\mathrm{supp} f \subseteq U_N \cap U_S$ and $f = 1$ on some open set $V_N \cap V_S \subseteq U_N \cap U_S$, where $V_N \subseteq U_N$, $V_S \subseteq U_S$ and $V_N \cup V_S = \mathcal{S}^n$. (The reason why we need to take the smaller sets V_N and V_S is that we obviously cannot find a bump function with support in $U_N \cap U_S$ which is 1 on $U_N \cap U_S$, because then it would be 1 on $\mathcal{S}^n$, violating the assumption on its support.)

Consider the $(k-1)$-form β on $\mathcal{S}^n$ defined by

$$\beta(p) = \begin{cases} \beta_N(p) & p \in V_N, \\ \big(\beta_S + \mathrm{d}(f\gamma)\big)(p) & p \in V_S. \end{cases}$$

This is well-defined because $f\gamma$ is well-defined on $\mathcal{S}^n$ and $f\gamma = \gamma$ on $V_N \cap V_S$, which means that

$$\beta_N = \beta_S + \mathrm{d}(f\gamma) = \beta_S + \mathrm{d}\gamma$$

on $V_N \cap V_S$. We also see that

$$\alpha = \mathrm{d}\beta_N = \mathrm{d}\beta$$

on V_N and

$$\alpha = \mathrm{d}\beta_S = \mathrm{d}(\beta_S + \mathrm{d}(f\gamma)) = \mathrm{d}\beta$$

on V_S, so $\alpha = \mathrm{d}\beta$ is exact and thus $[\alpha] = 0$. Therefore, $H^k(\mathcal{S}^n) = 0$ for $1 < k < n$ by induction, completing the proof. □

Remark. (Not examinable). This result can be proved much more quickly (still by induction) using a fundamental tool in algebraic topology called the *Mayer–Vietoris theorem*, which involves considering two open sets U and V and relating the cohomology of the union $U \cap V$ (in our case, the manifold we want to study), to the cohomology of U, V and $U \cap V$. The relation to the proof presented above should hopefully be clear.

We now return to the proof of Theorem 7.6 which we achieve in two steps. The first is a elementary lemma in $\mathbb{R}^n$. The proof is a bit long, but the argument is certainly elementary.

Lemma 7.8. *Let $I^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| < 1 \text{ for all } i\}$ and let ω be an n-form on $\mathbb{R}^n$ with support in I^n so that*

$$\int_{\mathbb{R}^n} \omega = 0.$$

Then, there exists an $(n-1)$-form $\mathrm{d}\eta$ with support in I^n so that $\omega = \mathrm{d}\eta$.

Proof. **(Not examinable).** Since ω is an n-form on $\mathbb{R}^n$, we can write it as

$$\omega = a\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$$

for a function a with support in I^n so that

$$\int_{-\infty}^{\infty} a\mathrm{d}x_1 \ldots \mathrm{d}x_n = 0.$$

If we let

$$\eta = \sum_{j=1}^{n} (-1)^{j-1} b_j \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \wedge \mathrm{d}x_n$$

then $\mathrm{d}\eta = \omega$ if and only if

$$a = \sum_{j=1}^{n} \frac{\partial b_j}{\partial x_j}.$$

Therefore, the result will follow if we can such find functions b_j with support in I^n.

We now work by induction on n. For $n = 1$ we are saying that a has support in $I^1 = (-1, 1)$ and

$$\int_{-\infty}^{\infty} a\mathrm{d}x_1 = 0,$$

so we can just choose

$$b_1(x) = \int_{-\infty}^{x} a(x_1)\mathrm{d}x_1,$$

which clearly has support in $(-1, 1)$ since a does.

Now suppose that the result holds for $n - 1$ and define

$$A(x_1, \ldots, x_{n-1}) = \int_{-\infty}^{\infty} a(x_1, \ldots, x_{n-1}, x_n)\mathrm{d}x_n.$$

Then

$$\int_{\mathbb{R}^{n-1}} A(x_1, \ldots, x_{n-1})\mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} = \int_{\mathbb{R}^n} a(x_1, \ldots, x_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n = 0$$

so, by the inductive hypothesis, there exist functions $B_1, \ldots, B_{n-1}$ with support on I^{n-1} so that

$$A = \sum_{j=1}^{n-1} \frac{\partial B_j}{\partial x_j}.$$

We then want to find our functions b_j with support on I^n, so it makes sense to just separate variables and choose a smooth function b with support in I^1 and define

$$b_j(x_1, \ldots, x_n) = B_j(x_1, \ldots, x_{n-1})b(x_n).$$

We see that

$$\sum_{j=1}^{n-1} \frac{\partial b_j}{\partial x_j} = \sum_{j=1}^{n-1} \frac{\partial B_j}{\partial x_j} b = Ab.$$

Hence, if we let

$$b_n = \int_{-\infty}^{x_n} a - Ab \, \mathrm{d}x_n$$

we see that

$$\sum_{j=1}^{n} \frac{\partial b_j}{\partial x_j} = \sum_{j=1}^{n-1} \frac{\partial b_j}{\partial x_j} + \frac{\partial b_n}{\partial x_n} = Ab + a - Ab = a$$

as required, so we just need that b is chosen so that b_n has support in I^n. Since $a - Ab$ has support in I^n, it just means that we need that for all $x > 1$,

$$\int_{-\infty}^{x} a - Ab \, \mathrm{d}x_n = 0.$$

However, we see that for $x > 1$, since A is independent of x_n,

$$\int_{-\infty}^{x} a - Ab \, \mathrm{d}x_n = \int_{-\infty}^{\infty} a - Ab \, \mathrm{d}x_n = \int_{-\infty}^{\infty} a \, \mathrm{d}x_n - A \int_{-\infty}^{\infty} b \, \mathrm{d}x_n.$$

Therefore, if we choose b so that

$$\int_{-\infty}^{\infty} b \, \mathrm{d}x_n = 1$$

we have that

$$\int_{-\infty}^{\infty} a - Ab \, \mathrm{d}x_n = \int_{-\infty}^{\infty} a \, \mathrm{d}x_n - A = 0$$

by definition of A. The result thus follows by induction. □

The second step is to choose a finite cover of our compact manifold by copies of I^n and use connectedness to get the result.

Proof of Theorem 7.6. **(Not examinable).** We know that since M is compact, connected, and orientable, there exists a volume form Ω so that $\int_M \Omega = v > 0$. By taking constant multiples of Ω we therefore see that

$$[\omega] \mapsto \int_M \omega$$

is surjective. We also see that if $\mathrm{d}\eta \in \mathcal{E}^n(M)$, then $\int_M \mathrm{d}\eta = 0$ by Stokes Theorem, so to show that the map is injective we need to show that

$$\int_M \omega = 0 \quad \Rightarrow \quad \omega = \mathrm{d}\eta.$$

To that end, suppose that ω is an n-form which integrates to 0. The idea is to reduce to the case of the previous lemma. Therefore, suppose we take an orientation with charts (U_i, φ_i) so that $\varphi_i(U_i) \subseteq I^n$ for all i. Since M is compact, there is a finite number of charts, say $(U_1, \varphi_1), \ldots (U_N, \varphi_N)$ which cover M, which again gives an orientation. As is now standard practice, we take a partition of unity $\{f_j : j \in \mathbb{N}\}$ subordinate to this finite cover (which can now be taken to be a finite number of functions since the cover is finite). We now claim that, for all j, there exists $\mathrm{d}\zeta_j$ so that

$$\alpha_j = f_j \omega + \mathrm{d}\zeta_j$$

is supported in U_1. (We picked U_1 just for convenience, but any fixed choice of chart would do.)

Assuming this claim, we see that if we let $\zeta = \sum_{j=1}^{\infty} \zeta_j$ then

$$\alpha = \sum_{j=1}^{\infty} \alpha_j = \omega + \mathrm{d}\zeta$$

has support in U_1 and

$$\int_{U_1} \alpha = \int_M \alpha = \int_M \omega + \int_M \mathrm{d}\zeta = 0$$

by assumption and Stokes Theorem. Therefore,

$$\int_{\varphi_1(U_1)} (\varphi_1^{-1})^*\alpha = \int_{\mathbb{R}^n} (\varphi_1^{-1})^*\alpha = 0$$

By Lemma 7.8 we see that $(\varphi_1^{-1})^*\alpha = \mathrm{d}\beta$ for β with support in $\varphi_1(U_1)$ and hence $\alpha = \mathrm{d}\gamma$ for some γ with support in U_1. We deduce that

$$\mathrm{d}\gamma = \alpha = \omega + \mathrm{d}\zeta$$

and so $\omega = \mathrm{d}(\gamma - \zeta)$ as required.

To finish the proof, we need just to prove the claim about the existence of the $\mathrm{d}\zeta_j$. This is where connectedness comes in. Suppose that $f_j\omega$ has support in U_m. By connectedness we must have $U_1, \ldots, U_m$ (relabelling if necessary) so that $U_k \cap U_{k-1} \neq \emptyset$ for all k. Choose any form β_0 with support in $U_m \cap U_{m-1}$ so that

$$\int_{U_m} \beta_0 = \int_{U_m} f_j\omega;$$

this is possible because there is a volume form on $U_m \cap U_{m-1}$. Then $f_j\omega - \beta_0$ has support in U_m and

$$\int_{\varphi_m(U_m)} (\varphi_m^{-1})^*(f_j\omega - \beta_0) = 0.$$

By Lemma 7.8 there exists η_0 with support in U_m so that

$$f_j\omega - \beta_0 = \mathrm{d}\eta_0.$$

We now repeat the process starting with β_0 and choose a form β_1 with support in $U_{m-2} \cap U_{m-1}$ so that

$$\int_{U_{m-1}} \beta_1 = \int_{U_{m-1}} \beta_0.$$

There then exists η_1 with support in U_{m-1} so that

$$\beta_0 - \beta_1 = \mathrm{d}\eta_1.$$

Continuing we find forms β_k with support in $U_{m-k-1} \cap U_{m-k}$ and η_k with support in U_{m-k} so that

$$\beta_{k-1} - \beta_k = \mathrm{d}\eta_k,$$

terminating in β_{m-1} and η_{m-1} with support in U_1. We may then write

$$\begin{aligned} f_j\omega &= \beta_0 + \mathrm{d}\eta_0 \\ &= \beta_0 - \beta_1 + \beta_1 + \mathrm{d}\eta_0 \\ &= \sum_{k=1}^{m-1} (\beta_{k-1} - \beta_k) + \beta_{m-1} + \mathrm{d}\eta_0 \\ &= \sum_{k=0}^{m-1} \mathrm{d}\eta_k + \beta_{m-1}. \end{aligned}$$

Therefore, if we set $\zeta_j = -\sum_{k=0}^{m-1} \eta_k$ we see that

$$f_j\omega + \mathrm{d}\zeta_j = \beta_{m-1}$$

has support in U_1 as required. □

We can see an important topological application of our theorem about de Rham cohomology of spheres, which is often called the Hairy Ball Theorem.

Theorem 7.9. *Every vector field on $\mathcal{S}^{2n}$ vanishes somewhere.*

Proof. Suppose not, so we have a vector field X on $\mathcal{S}^{2n}$ which is nowhere vanishing. Viewing $X(p) \in T_p\mathcal{S}^{2n}$ as a vector in $\mathbb{R}^{2n+1}$ orthogonal to p, we therefore have a smooth map $h : \mathcal{S}^{2n} \times [0,1] \to \mathcal{S}^{2n}$ given by

$$h(p,t) = p\cos(\pi t) + \frac{X(p)}{|X(p)|}\sin(\pi t).$$

Notice that

$$h_0(p) = h(p,0) = p \quad \text{and} \quad h_1(p) = h(p,1) = -p,$$

so $h_0 = \mathrm{id}$ and $h_1 = -\mathrm{id}$. Thus, $h_0^* = \mathrm{id}$ and $h_1^* = (-1)^{2n+1}\,\mathrm{id} = -\mathrm{id}$ on $H^{2n}(\mathcal{S}^{2n})$: this follows from our earlier example, where we saw that $-\mathrm{id}$ is orientation reversing on $\mathcal{S}^{2n}$.

However, Theorem 7.4 states that $\mathrm{id} = h_0^* = h_1^* = -\mathrm{id}$ on $H^{2n}(\mathcal{S}^{2n})$, which would force $H^{2n}(\mathcal{S}^{2n}) = 0$, contradicting Theorem 7.7. □

Example. Let M be a compact, connected, oriented 4-dimensional manifold. Then for any $[\alpha], [\beta] \in H^2(M)$ we see that

$$Q([\alpha],[\beta]) = [\alpha] \cup [\beta] \in H^4(M) = \mathbb{R}.$$

We see immediately that the map $Q : H^2(M) \times H^2(M) \to \mathbb{R}$ is symmetric since

$$[\beta] \cup [\alpha] = (-1)^4[\alpha] \cup [\beta] = [\alpha] \cup [\beta],$$

and bilinear, so Q defines a quadratic form on $H^2(M)$. This quadratic form is an important object in the study of geometry and topology of 4-dimensional manifolds.

7.3 Degree

For the whole of this section, suppose that we have a compact, connected, oriented, n-dimensional manifold M.

By Theorem 7.6, there exists an n-form ω_M so that

$$\int_M \omega_M = 1,$$

and ω_M is unique up to the addition of exact forms, i.e. $[\omega_M] \in H^n(M)$ is unique.

We want to use this observation to introduce a key notion: the degree of a map between compact, connected, oriented n-dimensional manifolds.

Definition 7.10. Let M and N be compact, connected, oriented, n-dimensional manifolds, and let ω_M and ω_N be n-forms on M and N whose integral is 1. Let $f : M \to N$ be a smooth map. By Theorem 7.6, there exists a real number $\deg f$ so that

$$f^*[\omega_N] = \deg f[\omega_M]$$

since $H^n(M) = H^n(N) = \mathbb{R}$. Equivalently,

$$\int_M f^*\omega_N = \deg f \int_M \omega_M = \deg f.$$

The number $\deg f$ is called the degree of f.

Example. If $f = \mathrm{id} : M \to M$ is the identity map, then $f^*\omega_M = \mathrm{id}^*\,\omega_M = \omega_M$ so $\deg \mathrm{id} = 1$.

Example. Let $f = -\operatorname{id} : \mathcal{S}^n \to \mathcal{S}^n$. Then we saw that $f^*\omega_{\mathcal{S}^n} = (-\operatorname{id})^*\omega_{\mathcal{S}^n} = (-1)^{n+1}\omega_{\mathcal{S}^n}$. Therefore, $\deg(-\operatorname{id}) = (-1)^{n+1}$.

Example. Fix $c \in M$ and let $f : M \to M$ be given by $f(p) = c$ for all $p \in M$, i.e. f is a constant map. Then $f^* = 0$ so, in particular, $f^*\omega_M = 0$. Hence, $\deg f = 0$ in this case if M is n-dimensional for $n > 0$.

We make two elementary observations about the degree.

Lemma 7.11. *Let $f : M \to N$ be a smooth map between compact, connected, oriented n-dimensional manifolds. Then*
$$\int_M f^*\omega = \deg f \int_N \omega$$
for all n-forms ω on N.

Proof. Let ω be an n-form on N. Then $H^n(N) = \mathbb{R}$, there is constant c so that $[\omega] = c[\omega_N]$. As we saw earlier, this means that
$$\int_N \omega = c\int_N \omega_N = c.$$
Moreover, $f^*[\omega] = cf^*[\omega_N] = c\deg f[\omega_M]$ by definition, and thus
$$\int_M f^*\omega = c\deg f \int_M \omega_M = c\deg f = \deg f \int_N \omega,$$
as required. □

Lemma 7.12. *Let $h : M \times [0,1] \to N$ be a smooth map, where M, N are compact, connected, oriented n-dimensional manifolds, and let $h_t(p) = h(p,t)$. Then $\deg h_0 = \deg h_1$.*

Proof. By Theorem 7.4, $h_0^* = h_1^*$ as maps from $H^n(N)$ to $H^n(M)$. Therefore,
$$\deg h_0[\omega_M] = h_0^*[\omega_N] = h_1^*[\omega_N] = \deg h_1[\omega_M].$$
The result follows since $[\omega_M] \neq 0$. □

We saw in our examples that $\deg f$ turned out to be an integer. We now show that this is always the case.

Theorem 7.13. *Let $f : M \to N$ be a smooth map between compact, connected, oriented n-dimensional manifolds. Let $c \in N$ be a regular value of f, i.e. so that $\mathrm{d}f_p : T_pM \to T_cN$ is surjective for all $p \in f^{-1}(c)$. Then*
$$\deg f = \sum_{p \in f^{-1}(c)} \operatorname{sgn}\det(\mathrm{d}f_p),$$
where $\operatorname{sgn}\det(\mathrm{d}f_p)$ denotes the sign of the determinant of $\mathrm{d}f_p$.

In particular, $\deg f$ is an integer and $\deg f = 0$ if f is not surjective.

Remark. (Not examinable.) As we noted before, regular values have full measure in N and so, in particular, the set of regular values is non-empty (though it should be noted that the regular value condition is vacuous if the point in N is not in the image of the map).

Proof. The Regular Value Theorem (Theorem 2.8) implies that $f^{-1}(c)$ is a 0-dimensional manifold, which must be compact (because M is compact and $f^{-1}(c)$ is closed), and so consists of only finitely many points $p_1, \ldots, p_m$.

Moreover, f is a local diffeomorphism at p_i for all i, so there exist disjoint connected open sets $U_i \ni p_i$ contained in coordinate charts in M and open sets $V_i \ni c$ contained in coordinate charts in N so that $f : U_i \to V_i$ is a diffeomorphism. Letting $V = \cap_{i=1}^m V_i$, we can make the U_i smaller so that $f : U_i \to V$ is a diffeomorphism.

Note that $f : U_i \to V$ is orientation preserving (respectively reversing) if and only if $\operatorname{sgn}\det(\mathrm{d}f_p)$ is positive (respectively negative). Therefore, if we choose an n-form ω_V on N with support on V so that

$$\int_N \omega_V = \int_V \omega_V = 1,$$

we see that

$$\int_{U_i} f^*\omega_V = \operatorname{sgn}\det(\mathrm{d}f_{p_i})\int_V \omega_V = \operatorname{sgn}\det(\mathrm{d}f_{p_i})$$

by using the coordinate invariance of integration of n-forms on $\mathbb{R}^n$ (and the orientations on M and N, which were used to define $\operatorname{sgn}\det(\mathrm{d}f_{p_i})$). Since ω_V has support in V, $f^*\omega_V$ has support in $\cup_{i=1}^m U_i$ (which is a disjoint union). Hence, by Lemma 7.11,

$$\begin{aligned}\deg f &= \deg f \int_N \omega_V\\ &= \int_M f^*\omega_V\\ &= \sum_{i=1}^m \int_{U_i} f^*\omega_V\\ &= \sum_{i=1}^m \operatorname{sgn}\det(\mathrm{d}f_{p_i}),\end{aligned}$$

as required. □

We now want to briefly describe some applications of degree through examples.

Example. View $\mathcal{S}^1 \subseteq \mathbb{C}$ as the unit complex numbers. Let $g : \mathcal{S}^1 \to \mathbb{C}$ be a smooth map and let $w \in \mathbb{C}$ such that $g^{-1}(w) = \emptyset$. Consider $f : \mathcal{S}^1 \to \mathcal{S}^1$ given by

$$f(z) = \frac{g(z) - w}{|g(z) - w|}.$$

Then the degree of f is the *winding number* of the closed curve g around w.

Example. In the setting of the previous example, if we take $g(z) = z^k$ for some integer $k > 0$ and $w = 0$, we see that $|g(z)| = 1$ in $\mathcal{S}^1$ (i.e. the image of g is the unit circle) and so $f = g$. In particular, 1 lies in $f(\mathcal{S}^1) = \mathcal{S}^1$. We see that $f^{-1}(1)$ consists of k points which are the kth roots of unity. We also note that $\mathrm{d}f_1$ is multiplication by k, which means that $\det(\mathrm{d}f_1) = k$ and thus $\operatorname{sgn}\det(\mathrm{d}f_1) > 0$. We deduce that $\operatorname{sgn}\det(\mathrm{d}f_z) > 0$ for all $z \in f^{-1}(1)$ (since we can rotate the circle anticlockwise to map 1 to any other kth root of unity, which is an orientation preserving diffeomorphism). We deduce that

$$\deg f = k,$$

that is, the winding number of g about the origin is k (which intuitively makes sense). Replacing k by $-k$ we obtain a curve with winding number $-k$ (which, again, intuitively makes sense). Moreover, taking $k = 0$ gives a constant map $g(z) = 1 = f(z)$, which has degree 0.

Example. Sticking with the complex numbers theme, identify $\mathcal{S}^2$ with $\mathbb{C} \cup \{\infty\}$ and consider the map $f : \mathcal{S}^2 \to \mathcal{S}^2$ given by a monomial of degree k, so

$$f(z) = z^k + a_{k-1}z^{k-1} + \ldots + a_0$$

and set $f(\infty) = \infty$. Then if we let $h : \mathcal{S}^2 \times [0,1] \to \mathcal{S}^2$ be given by

$$h(z,t) = z^k + t(a_{k-1}z^{k-1} + \ldots + a_0)$$

and $h(\infty, t) = \infty$, we see that $h_0(z) = h(z, 0) = z^k$ and $h_1(z) = h(z, 1) = f(z)$. We saw that $\deg f = \deg(z^k)$, and it follows from the previous example that $\deg(z^k) = k$.

Hence, if $k > 0$, $\deg f = k > 0$ and in particular $f(z) = w$ has a solution for any $w \in \mathbb{C}$ (as f must be surjective). We have thus proved the *Fundamental Theorem of Algebra*.

Example. Let $g_1, g_2 : \mathcal{S}^1 \to \mathbb{R}^3$ be two smooth maps and suppose that $g_1(\mathcal{S}^1) \cap g_2(\mathcal{S}^1) = \emptyset$. We therefore have (whenever the maps g_1, g_2 are immersions) two disjoint loops in $\mathbb{R}^3$. Define

$$f(p, q) = \frac{g_1(p) - g_2(q)}{|g_1(p) - g_2(q)|},$$

which makes sense since $g_1(p) \neq g_2(q)$ and gives a map $f : \mathcal{S}^1 \times \mathcal{S}^1 \to \mathcal{S}^2$.

The degree of f is called the *linking number* of the curves g_1 and g_2. This measures how much the two curves are "linked".

We now give several examples of linking numbers.

Example. If we take $g_1 : \mathcal{S}^1 \to \mathbb{C} \subseteq \mathbb{R}^3$ (where $\mathbb{C}$ is identifed with the plane where $x_3 = 0$) and $g_2(q) = a$ for all $q \in \mathcal{S}^1$ for some $a \in \mathbb{C} \setminus g_1(\mathcal{S}^1)$, then the linking number of g_1 and g_2 is just the winding number of g_1 around a.

Example. If we take $g_1, g_2 : \mathcal{S}^1 \to \mathbb{R}^3$ given by

$$g_1(x_1, x_2) = (x_1 + 2, x_2, 0) \quad \text{and} \quad g_2(x_1, x_2) = (x_1 - 2, x_2, 0),$$

so we have unit circles in the plane $x_3 = 0$ with centres at $(2, 0, 0)$ and $(-2, 0, 0)$. Then f can never equal $(0, 0, 1)$ for example, and thus f is not surjective. Hence $\deg f = 0$ and the linking number of g_1 and g_2 is 0. This makes sense as we can just pull the two circles apart.

Example. A non-trivial example is given by $g_1, g_2 : \mathcal{S}^1 \to \mathbb{R}^3$ with

$$g_1(x_1, x_2) = (x_1 - 1, x_2, 0) \quad \text{and} \quad g_2(x_1, x_2) = (x_1, 0, x_2).$$

Then

$$f(x_1, x_2, y_1, y_2) = \frac{(x_1 - y_1 - 1, x_2, -y_2)}{\sqrt{(x_1 - y_1 - 1)^2 + x_2^2 + y_2^2}}.$$

We see that $f^{-1}(0, 1, 0) = \{(0, 1, -1, 0)\}$ and from this it is not hard to check that $\deg f = 1$ (it has to be 1 or -1). We can see visually that these two circles are linked.

Example. (Not examinable.) Let X be a vector field on $\mathbb{R}^n$ and suppose that X has an isolated zero at p (i.e. $X(p) = 0$ and $X(q) \neq 0$ for all $q \neq p$ near p): p is often called a singularity of X. Then for $\epsilon > 0$ sufficiently small, we may define $f : \mathcal{S}^{n-1} \to \mathcal{S}^{n-1}$ by

$$f(q) = \frac{X(p + \epsilon q)}{|X(p + \epsilon q)|},$$

where we view $X(p + \epsilon q)$ as a vector in $\mathbb{R}^n$. The degree of f is called the *index* of X at p:

$$\deg f = I(X, p).$$

(It is not hard to see that this is independent of the choice of ϵ small.) Informally, this might be called "the order of vanishing of X at p" (up to sign), just as the degree of z^k we saw before was k.

Now, suppose we have a vector field X on an oriented n-dimensional manifold M with only finitely many zeros $p_1, \ldots, p_m$. If $\{(U_i, \varphi_i) : i \in I\}$ defines the orientation so that each p_j lies in only one chart (U_j, φ_j), we define the index of X to be

$$I(X) = \sum_{j=1}^{m} I((\varphi_j)_*(X), \varphi_j(p_j)).$$

Finally, let M be a compact n-dimensional manifold. We define the *Euler number* of M to be

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M).$$

If M is also oriented, then for any vector field X on M with only isolated zeros (which are necessarily finite in number), we have that

$$\chi(M) = I(X).$$

This is the *Poincaré–Hopf Theorem*.

Example. (Not examinable.) If M is compact and oriented with $\chi(M) \neq 0$, then every vector field on M must have zeros (since its index is non-zero by the Poincaré–Hopf Theorem). In particular, since $\chi(\mathcal{S}^{2n}) = 1 + (-1)^{2n} = 2$ by Theorem 7.7, a vector field on $\mathcal{S}^{2n}$ must have zeros which is the Hairy Ball Theorem.

8 Riemannian manifolds

In the history of modern differential geometry, Riemannian geometry was invented first (by Riemann in his habilitation thesis), but to put it on a firm foundational footing the theory of manifolds had to be developed. Riemannian manifolds are very important in the study of geometry and topology, but here we will just give an introduction to two fundamental notions in the subject: isometries and geodesics.

8.1 Definition

The idea is to have a notion of a way of measuring distance which varies from point to point. Again, we begin with a fake definition.

Fake definition: A Riemannian metric g on M is a smooth choice of positive definite inner product on each tangent space, i.e. for each $p \in M$ we have a symmetric bilinear map $g_p : T_pM \times T_pM \to \mathbb{R}$ which is positive definite.

Given the theory we have developed, we can quickly make the precise definition.

Definition 8.1. A *Riemannian metric* g on a manifold M is a positive definite section of S^2T^*M.

A *Riemannian manifold* (M, g) is a manifold M endowed with a Riemannian metric g.

We will prove shortly that every manifold can be given a Riemannian metric, so any manifold is a Riemannian manifold, but of course the metric is not unique, and the geometry of the Riemannian manifold can vary wildly even though one has the same underlying manifold. For a simple example, consider the sphere, ellipsoid and dumbbell, which are all diffeomorphic to $\mathcal{S}^2$, but clearly have very different geometries.

8.2 Examples

Let us try to understand what Riemannian metrics are. As usual, it suffices to do this locally. We have a basis for $S^2T^*\mathbb{R}^n$ given by $\mathrm{d}x_i\mathrm{d}x_j$ where

$$\mathrm{d}x_i\mathrm{d}x_j(\partial_k, \partial_l) = \mathrm{d}x_i\mathrm{d}x_j(\partial_l, \partial_k) = \begin{cases} 1 & \text{if } i = j = k = l, \\ \frac{1}{2} & \text{if } i \neq j \text{ and either } i = k, j = l \text{ or } i = l, j = k, \\ 0 & \text{otherwise.} \end{cases}$$

(Here $\mathrm{d}x_i\mathrm{d}x_j$ is just $\mathcal{S}(\mathrm{d}x_i \otimes \mathrm{d}x_j)$, where $\mathcal{S}$ is the symmetrization map we saw earlier.) Therefore, any Riemannian metric on an open set in $\mathbb{R}^n$ can be written as

$$\sum_{i,j} g_{ij}\mathrm{d}x_i\mathrm{d}x_j$$

where g_{ij} is a positive definite symmetric matrix of functions.

Example. On $\mathbb{R}^n$, the Euclidean metric g_0 is given as

$$g_0 = \mathrm{d}x_1^2 + \ldots \mathrm{d}x_n^2.$$

Here, the matrix g_0 is identified with is just the identity matrix.

Example. Let $M \subseteq \mathbb{R}^n$. We can define a Riemannian metric on M by $g_p(X, Y) = g_0(X, Y)$, since if $X, Y \in T_pM$ then $X, Y \in T_p\mathbb{R}^n$. We call this the *induced metric* on M.

In the case where M is a surface in $\mathbb{R}^3$ then the induced metric is nothing other than the *first fundamental form* of M.

In particular we get that $\mathcal{S}^n$ has a Riemannian metric induced from the Euclidean metric on $\mathbb{R}^{n+1}$.

Example. Let H^n be the n-dimensional upper half-space and define the *hyperbolic metric* g on H^n by

$$g = \frac{\mathrm{d}x_1^2 + \ldots + \mathrm{d}x_n^2}{x_n^2}.$$

This metric plays an important role in geometry and topology.

Remark. Let X, Y be vector fields on (M, g). Then $g(X, Y)$ is a function, whose value at $p \in M$ is $g_p(X(p), Y(p))$.

8.3 Pullback and existence of Riemannian metrics

Just as for forms, we can pullback Riemannian metrics as follows.

Definition 8.2. Let $f : M \to N$ be smooth and let h be a Riemannian metric on N. We define the pullback f^*h of h by f as:

$$(f^*h)_p(X, Y) = h_{f(p)}\big(\mathrm{d}f_p(X), \mathrm{d}f_p(Y)\big)$$

for $p \in M$ and $X, Y \in T_pM$.

Now, we have the made the definition for any smooth map, but when is f^*h actually a Riemannian metric?

Proposition 8.3. *Let $f : M \to N$ be an immersion (so $\mathrm{d}f_p$ is injective for all $p \in M$) and let h be a Riemannian metric on N. Then $g = f^*h$ is a Riemannian metric on M.*

Remark. This is the content of our earlier example of the induced metric in the case where f is the inclusion map of M in $\mathbb{R}^n$.

Proof. Let $p \in M$ and let $X, Y \in T_pM$. Since h is symmetric and bilinear and smooth and f is smooth, we see that g is symmetric and bilinear and smooth, so we only need to check that it is positive definite.

We see that

$$g_p(X, X) = h_{f(p)}(\mathrm{d}f_p(X), \mathrm{d}f_p(X)) \geq 0$$

and $g_p(X, X) = 0$ if and only if $\mathrm{d}f_p(X) = 0$. But $\mathrm{d}f_p$ is injective so $\mathrm{d}f_p(X) = 0$ if and only if $X = 0$. Hence g is positive definite and thus g is a Riemannian metric. □

Example. If $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ is $f(r, \theta) = (r\cos\theta, r\sin\theta)$ then

$$\begin{aligned}
f^*g_0(\partial_r, \partial_r) &= g_0(f_*(\partial_r), f_*(\partial_r)) = 1, \\
f^*g_0(\partial_r, \partial_\theta) &= g_0(f_*(\partial_r), f_*(\partial_\theta)) = 0, \\
f^*g_0(\partial_\theta, \partial_\theta) &= g_0(f_*(\partial_\theta), f_*(\partial_\theta)) = r^2,
\end{aligned}$$

so

$$f^*g_0 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2.$$

Example. Let $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ define local coordinates on $\mathcal{S}^2$. The standard induced Riemannian metric g on $\mathcal{S}^2$ is determined in the coordinates (θ, ϕ) by

$$f^*g(\partial_\theta, \partial_\theta) = g_0(f_*\partial_\theta, f_*\partial_\theta) = 1, \quad f^*g(\partial_\theta, \partial_\phi) = 0, \quad f^*g(\partial_\phi, \partial_\phi) = \sin^2\theta$$

so

$$f^*g = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2.$$

in these coordinates (which, again, should look familiar).

We now use the pullback construction to show the following.

Theorem 8.4. *Every manifold has a Riemannian metric.*

Proof. Let M be an n-dimensional manifold with an atlas $\{(U_i, \varphi_i) : i \in I\}$ and let g_0 be the standard Riemannian metric on $\mathbb{R}^n$. By Theorem 5.1 we have a partition of unity $\{f_j : j \in \mathbb{N}\}$ subordinate to the atlas. For each $j \in \mathbb{N}$ there exists $i(j) \in I$ such that $\operatorname{supp} f_j \subseteq U_{i(j)}$. Let $(U_j, \varphi_j) = (U_{i(j)}, \varphi_{(i(j))})$.

On U_j, since φ_j is a diffeomorphism we can use Proposition 8.3 to give that $g_j = \varphi_j^* g_0$ is a Riemannian metric on U_j. Therefore $f_j g_j$ is smooth, symmetric and bilinear on U_j. We can therefore define $f_j g_j \in \Gamma(S^2 T^* M)$ by setting it to be zero when f_j is zero.

We now let

$$g = \sum_{j=1}^{\infty} f_j g_j.$$

This is a well-defined section of S^2T^*M because the locally finite property means that the sum is always finite in a neighbourhood of any given p in M. We now just need to show that it is positive definite.

Suppose that $p \in M$. Then

$$g_p(X, X) = \sum_{j=1}^{\infty} f_j(p)(g_j)_p(X, X) \geq 0$$

for all $X \in T_pM$ since $f_j \geq 0$ and $(g_j)_p(X, X) \geq 0$ as it is a Riemannian metric. Moreover, $g_p(X, X) = 0$ implies that $f_j(p)(g_j)_p(X, X) = 0$ for all $j \in \mathbb{N}$. Since $\sum_{j=1}^{\infty} f_j = 1$ we know that there exists $j \in \mathbb{N}$ such that $f_j(p) > 0$ and hence $(g_j)_p(X, X) = 0$. But g_j is a Riemannian metric so $X = 0$.

We conclude that g is positive definite and thus a Riemannian metric on M. $\square$

8.4 Forms and the Hodge star

As we know, if we have an inner product $\langle ., . \rangle$ on a finite-dimensional vector space V, we obtain an isomorphism between V and its dual V^* by $v \mapsto \langle v, . \rangle$, and so we can define an inner product $\langle ., . \rangle^*$ on V^*. If g_{ij} is the matrix of $\langle ., . \rangle$ with respect to a basis $\{v_1, \ldots, v_n\}$ of V, i.e.

$$g_{ij} = \langle v_i, v_j \rangle$$

then with respect to the dual basis $\{v^1, \ldots, v^n\}$ of V^*, where

$$v^i(v_j) = \delta_{ij},$$

we have the matrix of $\langle ., . \rangle^*$ is the inverse g^{ij} of g_{ij}, i.e.

$$g^{ij} = \langle v^i, v^j \rangle^*.$$

We can clearly do the same construction with the Riemannian metric, which enables us to make the following definition.

Definition 8.5. Let (M, g) be a Riemannian manifold. For each $p \in M$, we let $g_p^* : T_p^*M \times T_p^*M \to \mathbb{R}$ denote the inner product on T_p^*M induced by the inner product g_p on T_pM under the isomorphism

$$X \in T_pM \mapsto g_p(X, \cdot) \in T_p^*M.$$

We can also extend g^* to k-forms by defining what it does to decomposable forms and extending by linearity. We achieve this by demanding that for decomposable forms $\xi_1 \wedge \ldots \wedge \xi_k, \eta_1 \wedge \ldots \wedge \eta_k \in \Lambda^k T_p^* M$ we have

$$g_p^*(\xi_1 \wedge \ldots \wedge \xi_k, \eta_1 \wedge \ldots \wedge \eta_k) = \det g_p^*(\xi_i, \eta_j).$$

In particular, this defines an inner product

$$(\xi, \eta) \mapsto g^*(\xi, \eta)$$

for k-forms ξ, η on (M, g).

Remark. The notation g^* is not standard. In fact, the inner product on forms is often still denoted g which, while convenient, could be confusing.

Example. Consider $(\mathbb{R}^4, g_0)$ and let

$$\omega^+ = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 \quad \text{and} \quad \omega^- = \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_3 \wedge \mathrm{d}x_4.$$

We see that

$$g^*(\omega^+, \omega^+) = g^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2, \mathrm{d}x_1 \wedge \mathrm{d}x_2) + 2g^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2, \mathrm{d}x_3 \wedge \mathrm{d}x_4) + g^*(\mathrm{d}x_3 \wedge \mathrm{d}x_4, \mathrm{d}x_3 \wedge \mathrm{d}x_4).$$

We may also calculate

$$\begin{aligned} g^*(\mathrm{d}x_i \wedge \mathrm{d}x_j, \mathrm{d}x_k \wedge \mathrm{d}x_l) &= \det \begin{pmatrix} g^*(\mathrm{d}x_i, \mathrm{d}x_k) & g^*(\mathrm{d}x_i, \mathrm{d}x_l) \\ g^*(\mathrm{d}x_j, \mathrm{d}x_k) & g^*(\mathrm{d}x_j, \mathrm{d}x_l) \end{pmatrix} \\ &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \\ &= \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ -1 & \text{if } i = l \text{ and } j = k \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

as we would expect. Therefore,

$$g^*(\omega^+, \omega^+) = 1 + 1 = 2 = g^*(\omega^-, \omega^-).$$

However,

$$g^*(\omega^+, \omega^-) = g^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2, \mathrm{d}x_1 \wedge \mathrm{d}x_2) - g^*(\mathrm{d}x_3 \wedge \mathrm{d}x_4, \mathrm{d}x_3 \wedge \mathrm{d}x_4) = 0.$$

Therefore ω^+ and ω^- are orthogonal with length 2, with respect to g^*.

Definition 8.6. Given a Riemannian metric g on an oriented manifold M there is always a distinguished volume form Ω called the Riemannian volume form. The Riemannian volume form is the unique unit length volume form (i.e. $g^*(\Omega, \Omega) = 1$) compatible with the orientation.

If (U, φ) is a chart and $\{X_1, \ldots, X_n\}$ are the coordinate vector fields on (U, φ), then we can define

$$g_{ij} = g(X_i, X_j)$$

which we can view as functions on U or $\varphi(U)$. Then Ω satisfies

$$(\varphi^{-1})^*\Omega = \sqrt{\det(g_{ij})}\Omega_0.$$

If (M, g) is compact, we can integrate Ω over M and so define the volume of (M, g) to be

$$\mathrm{Vol}(M, g) = \int_M \Omega.$$

Example. The Riemannian volume form on $(\mathbb{R}^n, g_0)$ with its standard orientation is the standard volume form Ω_0.

Example. It is a worthwhile exercise to check, for example, that the volume forms we introduced before on $\mathcal{S}^1$ and $\mathcal{S}^2$ are the Riemannian volume forms (with respect to the induced metric) and give the expected answer for the volume of $\mathcal{S}^1$ and $\mathcal{S}^2$.

Using the Riemannian volume form allows us to define the following useful map on forms: the *Hodge star*.

Definition 8.7. Let (M, g) be an oriented n-dimensional Riemannian manifold and let Ω be the Riemannian volume form on M. There is a unique linear map $* : \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{n-k}T^*M)$, called the *Hodge star*, which satisfies

$$\xi \wedge *\eta = g^*(\xi, \eta)\Omega$$

for all k-forms ξ, η on M.

Given a k-form ξ, the $n-k$-form $*\xi$ is often called the Hodge dual of ξ. It is important to observe that $*^2 = (-1)^{k(n-k)}$ acting on k-forms.

Remark. It is straightforward to see that the Hodge star exists and is unique by linear algebra. If $\{e_1, \ldots, e_n\}$ is a positively oriented orthonormal basis of a vector space, so that

$$e_1 \wedge \ldots \wedge e_n$$

is the unit volume form, we see that

$$*(e_1 \wedge \ldots \wedge e_k) = e_{k+1} \wedge \ldots \wedge e_n.$$

In general

$$*(e_{i_1} \wedge \ldots \wedge e_{i_k}) = e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}$$

where $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is an even permutation of $(1, \ldots, n)$. Since $\{e_1, \ldots, e_n\}$ is a basis, this defines the Hodge star by linearity.

Example. On $(\mathbb{R}^3, g_0)$ with the standard orientation we see that

$$*\mathrm{d}x_1 = \mathrm{d}x_2 \wedge \mathrm{d}x_3, \quad *\mathrm{d}x_2 = \mathrm{d}x_3 \wedge \mathrm{d}x_1, \quad *\mathrm{d}x_3 = \mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

We also see that

$$*(\mathrm{d}x_2 \wedge \mathrm{d}x_3) = \mathrm{d}x_1, \quad *(\mathrm{d}x_3 \wedge \mathrm{d}x_1) = \mathrm{d}x_2, \quad *(\mathrm{d}x_1 \wedge \mathrm{d}x_2) = \mathrm{d}x_3.$$

Example. We see that on $(\mathbb{R}^4, g_0)$ with the standard orientation $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_4$ we have that

$$*(\mathrm{d}x_1 \wedge \mathrm{d}x_2) = \mathrm{d}x_3 \wedge \mathrm{d}x_4 \quad \text{and} \quad *(\mathrm{d}x_3 \wedge \mathrm{d}x_4) = \mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

Therefore, by linearity we see that for ω^+ and ω^- we saw earlier we have:

$$*\omega^+ = \omega^+ \quad \text{and} \quad *\omega^- = -\omega^-.$$

The fact that the Hodge star maps 2-forms to 2-forms in 4 dimension is of central importance in the geometry and topology of 4-manifolds.

8.5 Isometries and Killing fields

A natural question to ask is when two Riemannian manifolds are "the same". Clearly being diffeomorphic is not enough, since we can have many different Riemannian metrics on $\mathcal{S}^2$, for example. The correct notion is the obvious one we now give.

Definition 8.8. A smooth map $f : (M, g) \to (N, h)$ between Riemannian manifolds is an *isometry* if f is a diffeomorphism and $g = f^*h$. Clearly, the isometries on (M, g) form a group, in fact a subgroup of $\mathrm{Diff}(M)$, which we denote $\mathrm{Isom}(M, g)$.

Remark. (Not examinable). One can obviously define the notion of local isometry just like local diffeomorphism. These are important, just as local diffeomorphisms are important.

Example. The identity map $\mathrm{id} : (M, g) \to (M, g)$ is an isometry.

Example. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map so $f(x) = Ax$ where $A = (a_{ij}) \in M_n(\mathbb{R})$. Then f_* is multiplication by A so

$$f^*g_0(\partial_i, \partial_j) = g_0(f_*\partial_i, f_*\partial_j) = g_0(A\partial_i, A\partial_j) = g_0\left(\sum_{k=1}^n a_{ki}\partial_k, \sum_{l=1}^n a_{lj}\partial_l\right) = \sum_{k=1}^n a_{ki}a_{kj}$$

since $g_0(\partial_i, \partial_j) = \delta_{ij}$.

Thus $f^*g_0 = g_0$ if and only if $\sum_{k=1}^n a_{ki}a_{kj} = \delta_{ij}$, i.e. $A^{\mathrm{T}}A = I$, so $A \in \mathrm{O}(n)$.

Example. Notice that if $a \in \mathbb{R}^n$ and we define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = x + a$, then $f^* = \mathrm{id}$ so f is an isometry of $(\mathbb{R}^n, g_0)$.

Example. Combining the previous two examples (modulo the fact that you need to prove isometries of $(\mathbb{R}^n, g_0)$ are affine transformations), we have $\mathrm{Isom}(\mathbb{R}^n, g_0) = \mathrm{O}(n) \ltimes \mathbb{R}^n$.

Example. Clearly, $\mathrm{Isom}(S^n, g) = \mathrm{O}(n+1)$ for the standard round metric by our earlier discussion (since this is the subgroup of the isometry group of $\mathbb{R}^{n+1}$ which preserves the n-sphere).

Just as for forms we can define the Lie derivative of a Riemannian metric which leads to a distinguished class of vector fields.

Definition 8.9. If $X \in \Gamma(TM)$ and g is a Riemannian metric on M then

$$\mathcal{L}_X g(p) = \lim_{t\to 0} \frac{(\phi_t^X)^* g_{\phi_t^X(p)} - g_p}{t},$$

where ϕ_t^X defines the flow of X near p.

We call vector fields X such that $\mathcal{L}_X g = 0$ *Killing fields.*

Example. We see that if X is a vector field on (M, g) so that $(\phi_t^X)^*g = g$, then X is a Killing field.

Example. We see that the flow of ∂_i is a translation, so $(\phi_t^{\partial_i})^*g_0 = g_0$. Hence, $\mathcal{L}_{\partial_i} g_0 = 0$.

Example. The vector fields

$$X_1 = x_2\partial_3 - x_3\partial_2, \quad X_2 = x_3\partial_1 - x_1\partial_3, \quad X_3 = x_1\partial_2 - x_2\partial_1$$

on $\mathbb{R}^3$ define flows which are rotations around the x_1, x_2, x_3-axes respectively, which are again isometries, so $\mathcal{L}_{X_i} g_0 = 0$ for $i = 1, 2, 3$.

Example. If we let $X = x_1\partial_1 + x_2\partial_2$ on $\mathbb{R}^2$ then the integral curves are defined by $x_i' = x_i$ which means $x_i(t) = x_i(0)e^t$. Therefore, the flow of X is $\phi_t^X(p) = e^t p$, which means

$$(\phi_t^X)^*g_0(Y, Z) = g_0((\phi_t^X)_*Y, (\phi_t^X)_*Z) = g_0(e^tY, e^tZ) = e^{2t}g_0(Y, Z).$$

Therefore,

$$\mathcal{L}_X g_0 = \lim_{t\to 0} \frac{e^{2t} - 1}{t} g_0 = \frac{d}{dt}(e^{2t})|_{t=0} g_0 = 2g_0.$$

Hence X is not a Killing field. In fact, in polar coordinates, $X = r\partial_r$.

8.6 Geodesics

We now wish to discuss one of the fundamental objects in Riemannian geometry, namely geodesics. We start by defining the length of a curve.

Definition 8.10. The *length* of a curve $\alpha : [0, L] \to M$ in (M, g) is

$$L(\alpha) = \int_0^L |\alpha'(t)|\mathrm{d}t = \int_0^L \sqrt{g\big(\alpha'(t), \alpha'(t)\big)}\mathrm{d}t.$$

We can always parametrize α so that $|\alpha'| = 1$, by defining a new parameter s (which is 0 when $t = 0$) by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\alpha'(t)|.$$

The parameter s is called arclength, and we will say that α is then parametrized by arclength. In this case, $L(\alpha) = L$.

Example. On $\mathbb{R}^2$, this formula for length just recovers the usual one: if $\alpha(t) = (x_1(t), x_2(t))$ then

$$L(\alpha) = \int_0^L \sqrt{(x_1')^2 + (x_2')^2}\mathrm{d}t.$$

So, for example, if $0 < a < b$ then the line from $(0, a)$ to $(0, b)$ given by $\alpha : [0, 1] \to \mathbb{R}^2$ where $\alpha(t) = (0, a(1 - t) + bt)$ has

$$L(\alpha) = \int_0^1 \sqrt{(b-a)^2}\mathrm{d}t = \int_0^1 b - a\,\mathrm{d}t = b - a.$$

Example. If instead we use the hyperbolic metric on the upper half-plane then we get

$$L(\alpha) = \int_0^L \frac{\sqrt{(x_1')^2 + (x_2')^2}}{x_2}\mathrm{d}t,$$

so now the same curve from $(0, a)$ to $(0, b)$ in the previous example has length

$$L(\alpha) = \int_0^1 \frac{b-a}{a + (b-a)t}\mathrm{d}t = \big[\log(a + (b-a)t)\big]_0^1 = \log\left(\frac{b}{a}\right).$$

In particular notice now that the length becomes infinite as $a \to 0$.

Remark. (Not examinable). We can naturally view (M, g) as a metric space by defining the metric $d(p, q)$ for $p, q \in M$ to be the infimum of $L(\alpha)$ over all curves from p to q.

The length defines a functional on curves in (M, g) with fixed endpoints. This functional enables us to define geodesics.

Definition 8.11. A curve $\gamma : [0, L] \to M$ is a *geodesic* in (M, g) if γ is a critical curve for length amongst all curves $\alpha : [0, L] \to M$ with $\alpha(0) = \gamma(0)$ and $\alpha(L) = \gamma(L)$.

Remark. (Not examinable). There are several definitions for geodesics, which are equivalent, and we will see another one shortly. It is not obvious from the definition, but any geodesic γ must have $|\gamma'|$ be constant.

We now have a very useful way to actually calculate geodesics. The proof is an easy calculation and we omit it, and besides we will see this formula in the Riemannian Geometry course.

Proposition 8.12. *Let (U,φ) be a chart on (M,g) and write*

$$(\varphi^{-1})^*g=\sum_{i,j}g_{ij}\mathrm{d}x_i\mathrm{d}x_j.$$

On $\varphi(U)$ define

$$L=\frac{1}{2}\sum_{i,j}g_{ij}x_i'x_j'.$$

Then γ given by $\varphi\circ\gamma=(x_1,\ldots,x_n)$ is a geodesic if and only if, for all k,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x_k'}\right)-\frac{\partial L}{\partial x_k}=0.$$

We now want to use Proposition 8.12 to calculate geodesics.

Example. For $\mathbb{R}^n$, we see that

$$L=\frac{1}{2}\sum_{i=1}^n(x_i')^2$$

so

$$\frac{\partial L}{\partial x_i'}=x_i'\quad\text{and}\quad\frac{\partial L}{\partial x_i}=0.$$

Hence, for $\gamma=(x_1,\ldots,x_n)$ to be a geodesic we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x_i'}\right)-\frac{\partial L}{\partial x_i}=x_i''=0$$

which define straight lines $x_k(t)=a_kt+b_k$.

Example. On the standard n-torus $T^n\subseteq\mathbb{R}^{2n}$, if we take $f(\theta_1,\ldots,\theta_n)=(\cos\theta_1,\sin\theta_1,\ldots,\cos\theta_n,\sin\theta_n)$ then $f^*g_0=g$ is given by

$$\mathrm{d}\theta_1^2+\ldots+\mathrm{d}\theta_n^2.$$

Hence, the equations defining geodesics are just

$$\theta_i''=0.$$

We deduce that $\theta_i=a_it+b_i$, so the geodesics are

$$\gamma(t)=(\cos(a_1t+b_1),\sin(a_1t+b_1),\ldots,\cos(a_nt+b_n),\sin(a_nt+b_n)),$$

the images of the straight lines in T^n.

Example. For S^2 we have $g=\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2$ so

$$L=\frac{1}{2}\big((\theta')^2+\sin^2\theta(\phi')^2\big).$$

Then

$$\frac{\partial L}{\partial\theta'}=\theta'\quad\text{and}\quad\frac{\partial L}{\partial\theta}=\sin\theta\cos\theta(\phi')^2$$

so the first geodesic equation is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\theta'}\right)-\frac{\partial L}{\partial\theta}=\theta''-\sin\theta\cos\theta(\phi')^2=0.$$

We also have

$$\frac{\partial L}{\partial\phi'}=\sin^2\theta\phi'\quad\text{and}\quad\frac{\partial L}{\partial\phi}=0$$

so the other geodesic equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\phi'}\right)-\frac{\partial L}{\partial\phi}=(\sin^2\theta\phi')'=\sin^2\theta\phi''+2\sin\theta\cos\theta\theta'\phi'=0.$$

We see that $\phi' = 0$ and $\theta'' = 0$ gives a normalised geodesic if $\theta' = 1$, which is

$$\gamma(t) = \big(\sin(t+\theta_0)\cos\phi_0, \sin(t+\theta_0)\sin\phi_0, \cos(t+\theta_0)\big)$$

with θ_0, ϕ_0 constant, called a great circle.

We want to understand geodesics from an alternative perspective now, relating them to integral curves, but now not on M itself, but rather on T^*M.

Definition 8.13. Let $\pi : T^*M \to M$ be the projection map. For $X \in T_{(p,\xi)}(T^*M)$ (where $\xi \in T_p^*M$) we see that $\mathrm{d}\pi_{(p,\xi)}(X) \in T_pM$, and so $\xi(\mathrm{d}\pi_{(p,\xi)}(X))$ makes sense. We may therefore define the *tautological* 1*-form* τ on T^*M by

$$\tau_{(p,\xi)}(X) = \xi\big(\mathrm{d}\pi_{(p,\xi)}(X)\big).$$

(In a suitable sense, τ at the point (p,ξ) is just the 1-form ξ acting on T_pM.)

We then define the *canonical* 2*-form* ω on T^*M by

$$\omega = -\mathrm{d}\tau.$$

We will see that ω is non-degenerate, in the sense that the map from vector fields to 1-forms on T^*M given by

$$X \mapsto i_X\omega$$

is an isomorphism.

Example. For $\mathbb{R}^n$ we have that $T^*\mathbb{R}^n = \mathbb{R}^{2n}$, so we let $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ be coordinates on $T^*\mathbb{R}^n$. We then see that the tautological 1-form on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ is just

$$\tau = \sum_{i=1}^n y_i \mathrm{d}x_i.$$

Therefore, the canonical 2-form is

$$\omega = -\mathrm{d}\tau = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

It is then straightforward to see that

$$X \mapsto i_X\omega$$

is an isomorphism between vector fields and 1-forms on $\mathbb{R}^n$.

Remark. On each chart of (U,φ) of M, T^*U is trivial, and thus the discussion in the example above applies to give a local description of τ and ω, and hence show that $X \mapsto i_X\omega$ is an isomorphism as claimed.

Remark. The canonical 2-form on T^*M is one of the key examples of a *symplectic form*: a closed 2-form which is non-degenerate.

Definition 8.14. Define a function H on T^*M by

$$H(p,\xi) = \frac{1}{2}g_p^*(\xi,\xi).$$

Since $\mathrm{d}H$ is a 1-form on T^*M therefore exists a unique vector field X_H on T^*M such that

$$i_{X_H}\omega = \mathrm{d}H,$$

where ω is the canonical 2-form on T^*M.

We say that the flow of X_H on T^*M is the *geodesic flow* of (M,g).

The reason for the definition is the following result.

Proposition 8.15. *If α is an integral curve of X_H on T^*M then $\gamma = \pi\circ\alpha$ is a geodesic on M.*

Proof. **(Not examinable).** We will work in a chart (U,φ) and have coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ on T^*U. Then, on T^*U,

$$H(x,y) = \frac{1}{2}\sum_{i,j} g^{ij}y_iy_j.$$

Hence,

$$\mathrm{d}H = \frac{1}{2}\sum_{i,j,k}\partial_k g^{ij}y_iy_j\mathrm{d}x_k + \sum_{i,j} g^{ij}y_i\mathrm{d}y_j.$$

We see therefore that, on T^*U,

$$X_H = \sum_{i,j} g^{ij}y_i\frac{\partial}{\partial x_j} - \frac{1}{2}\sum_{i,j,k}\partial_k g^{ij}y_iy_j\frac{\partial}{\partial y_k}.$$

We deduce that, locally, the integral curves $\alpha = (x_1,\ldots,x_n,y_1,\ldots,y_n)$ of X_H satisfy

$$x_k' = \sum_i g^{ik}y_i \quad\text{and}\quad y_k' = -\frac{1}{2}\sum_{i,j}\partial_k g^{ij}y_iy_j.$$

We then see that

$$y_k = \sum_i g_{ki}x_i',$$

so using this as a definition of y_k given $x_1,\ldots,x_n$ and substituting in the second equation, we see that

$$(\sum_i g_{ki}x_i')' + \frac{1}{2}\sum_{i,j,a,b} g_{ia}\partial_k g^{ij}g_{jb}x_a'x_b' = 0.$$

We see that this is nothing other than the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x_k'}\right) - \frac{\partial L}{\partial x_k} = 0$$

we saw earlier, once we notice that

$$\sum_{i,j} g_{ia}\partial_k g^{ij}g_{jb} = -\partial_k g_{ab},$$

i.e. that if A is a matrix then $\mathrm{d}A^{-1} = -A^{-1}\mathrm{d}AA^{-1}$ (a matrix version of $(f^{-1})' = -f^{-2}f'$). □

Remark. Notice that, in local coordinates, $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ we have that

$$X_H = \sum_{i=1}^n \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}$$

Hence, $\alpha(t) = (x_1(t),\ldots,x_n(t),y_1(t),\ldots,y_n(t))$ is an integral curve of X_H if and only if

$$x_i' = \frac{\partial H}{\partial y_i} \quad\text{and}\quad y_i' = -\frac{\partial H}{\partial x_i}$$

for $i = 1,\ldots,n$. This should be familiar as *Hamilton's equations*, which is why H is often called a *Hamiltonian*. Notice that the first of Hamilton's equations defines the y_i in terms of x_i'.

We will not pursue this line of thought much further, other than to observe two simple results, which enable us to sometimes completely solve the geodesic equations without doing any integration.

Proposition 8.16. *The function H is constant along any integral curve of X_H.*

Proof. We see that

$$X_H(H) = \mathcal{L}_{X_H}(H) = i_{X_H}\mathrm{d}H = \omega(X_H,X_H) = 0,$$

by definition. □

Proposition 8.17. *Suppose that on a chart (U, φ) on (M, g) the vector field X is a Killing field so that*

$$\varphi_*(X) = \sum_{i=1}^n a_i \partial_i.$$

Then the function f, given in the local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on T^*M by

$$\sum_{i=1}^n a_i y_i$$

is constant along any integral curve of X_H.

Proof. **(Not examinable).** Since X is a Killing field, we have that $\tilde{X}(H) = 0$ where $\tilde{X}$ is a natural lift of the vector field X to T^*M (using the local coordinates), as H is determined purely by the Riemannian metric. It is straightforward to see from our formulas above that $\tilde{X}(H) = 0$ means that $X_H(f) = 0$. $\square$

We now put this formalism to use.

Example. Let (H^2, g) be upper-half plane with the metric

$$g = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{x_2^2}$$

and consider an integral curve α of X_H.

Then the function H is

$$H = \frac{1}{2}x_2^2(y_1^2 + y_2^2) = c_0$$

is constant on α.

The metric g is translation invariant in the x_1-direction, so ∂_1 is a Killing field. Hence,

$$y_1 = c_1$$

for some constant c_1 on α.

Finally, we notice that if we multiply (x_1, x_2) by λ then the metric g does not change, so it is dilation invariant. Therefore $x_1\partial_1 + x_2\partial_2$ is a Killing field (remember, this was *not true* for the Euclidean metric). Hence,

$$x_1y_1 + x_2y_2 = c_2$$

for some constant c_2 on α.

We deduce that

$$(x_2y_2)^2 = (c_2 - c_1x_1)^2 = 2c_0 - c_1^2x_2^2.$$

Hence, we have that

$$(c_1x_1 - c_2)^2 + (c_1x_2)^2 = 2c_0,$$

which is a semi-circle centred on the x_1-axis if $c_1 \neq 0$. If $c_1 = 0$, then

$$0 = y_1 = \frac{x_1'}{x_2^2}$$

and thus x_1 is constant, which gives vertical half-lines. We have thus described all of the geodesics of (H^2, g).

8.7 Harmonic forms: not examinable

A Riemannian metric and an orientation allows us to introduce the following important notion, namely that of *harmonic* forms.

Definition 8.18. Let (M, g) be an oriented n-dimensional Riemannian manifold. We define the *codifferential* $\mathrm{d}^* : \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{k-1}T^*M)$ by

$$\mathrm{d}^* = (-1)^{nk+k+1} * \mathrm{d}*$$

where $*$ is the Hodge star. We say that a k-form ω is *co-closed* if

$$\mathrm{d}^*\omega = 0.$$

Notice that d^* is zero on functions by definition.

Remark. If (M, g) is a compact oriented Riemannian manifold , then we can define the L^2-inner product on k-forms by:

$$\langle \xi, \eta \rangle_{L^2} = \int_M \xi \wedge *\eta.$$

We then see that if $\partial M = \emptyset$, ω is a $(k-1)$-form and η is a k-form then

$$\langle \mathrm{d}\omega, \eta \rangle_{L^2} = \langle \omega, \mathrm{d}^*\eta \rangle_{L^2}.$$

In other words, d^* is the *adjoint* of the exterior derivative d.

Definition 8.19. Let (M, g) be an oriented n-dimensional Riemannian manifold. We say that a k-form ω is *harmonic* if it is closed and co-closed, i.e.

$$\mathrm{d}\omega = 0 \quad \text{and} \quad \mathrm{d}^*\omega = 0.$$

We let $\mathcal{H}^k(M)$ be the set of harmonic k-forms, which is naturally a vector space.

We can then state the following fundamental result, which is a version of the *Hodge theorem*.

Theorem 8.20. *Let (M, g) be a compact oriented Riemannian manifold. The map from $\mathcal{H}^k(M) \to H^k(M)$ given by $\omega \mapsto [\omega]$ is an isomorphism, so*

$$\dim \mathcal{H}^k(M) = \dim H^k(M).$$

In other words, there is a unique harmonic form in every de Rham cohomology class.

This result is very surprising, saying that one can obtain a topological invariant (the dimension of the de Rham cohomology groups) from the harmonic forms, which depend on the metric. This shows one of the many interactions between the metric and topology.