1. Say a projectile of mass $m$ is shot at time $t = 0$ from the origin with speed $V$ at an angle $\alpha$ to the horizontal. Throughout the motion the particle is acted on by gravity so that $\frac{d^2y}{dt^2} = -g$.
   (i) Write down the initial conditions $x(0), y(0), x'(0), y'(0)$.
   (ii) Determine the particle’s position vector $(x(t), y(t))$ at time $t$.
   (iii) Determine where the projectile lands (returns to ground level $y = 0$).
   (iv) What value of $\alpha$ maximises the distance travelled?
   (v) Show that
   $$\frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} \left( \frac{dy}{dt} \right)^2 + mgy = \frac{1}{2} m V^2$$
   throughout the motion.

2. If a spring, with spring constant $\alpha$, is stretched by an extension $x$, Hooke’s Law states that the force on the particle has magnitude $\alpha |x|$ towards the equilibrium. Thus whether the extension $x$ is positive (an extension) or negative (a compression) Newton’s Second Law gives
   $$m \frac{d^2x}{dt^2} = -\alpha x.$$
   Show that the general solution of this equation is
   $$x(t) = A \cos \omega t + B \sin \omega t,$$
   for constants $A$ and $B$, and where $\omega^2 = \alpha/m$. Show that
   $$\frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} \alpha x^2 = E$$
   is constant throughout the motion. What does the quantity $\frac{1}{2} \alpha x^2$ represent?

3. Say that the particle and spring in Question 2 lie on a rough table, so that there is a resistant frictional force of magnitude $\mu mg$ (coefficient of friction $\mu$) when the particle is in motion and we have
   $$m \frac{d^2x}{dt^2} = -\alpha x + \mu mg \quad \text{when } x \geq 0.$$
   Say that we have initially $x(0) = \varepsilon > 0$ and $x'(0) = 0$. What happens if $\mu \geq \alpha \varepsilon/(mg)$? Show that if
   $$\frac{\varepsilon \alpha}{2mg} < \mu < \frac{\varepsilon \alpha}{mg}$$
   then the particle comes to rest before its normal equilibrium position, and find the value of $x$ where this occurs.
   Show that
   $$\frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} \alpha x^2 = \frac{1}{2} \alpha \varepsilon^2 + \mu mg (x - \varepsilon),$$
   throughout the motion and explain the significance of the terms in this identity.

4. Consider a mass $m$ at the end of a light inextensible rod of length $l$ making small swings under gravity; let $\theta$ denote the angle the rod makes with the vertical.
   (i) Note that $\mathbf{r} = (l \sin \theta, -l \cos \theta)$. What do $d\mathbf{r}/dt$ and $d^2\mathbf{r}/dt^2$ equal?
   (ii) Use Newton’s Second Law to show that
   $$l \frac{d^2\theta}{dt^2} = -g \sin \theta,$$
   and find an expression for the tension in the rod.
   (iii) Show throughout the motion that $\frac{1}{2} ml^2 (d\theta/dt)^2 - mgl \cos \theta = E$ is constant.
   (iv) Say that the pendulum’s oscillations are small enough that the approximation $\sin \theta \approx \theta$ applies. Show that the pendulum’s swings have period $2\pi \sqrt{l/g}$.
   (v) More generally if the particle starts off with $\theta = \alpha, d\theta/dt = 0$, show that the oscillations have exact period
   $$4 \sqrt{ \frac{l}{2g} \int_{\alpha}^{0} \frac{d\theta}{\sqrt{\cos \theta - \cos \alpha}} }.$$