DYNAMICS 2 – Oscillations and Further Examples.

1. Say a projectile of mass m is shot at time $t = 0$ from the origin with speed V at an angle α to the horizontal. Throughout the motion the particle is acted on by gravity so that $\mathrm{d}^2y/\mathrm{d}t^2 = -g$.

(i) Write down the initial conditions $x(0), y(0), x'(0), y'(0)$.
(ii) Determine the particle's position vector $(x(t), y(t))$ at time t.
(iii) Determine where the projectile lands (returns to ground level $y = 0$).
(iv) What value of α maximises the distance travelled?
(v) Show that

$$\frac{1}{2}m\left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right) + mgy = \frac{1}{2}mV^2$$

throughout the motion.

2. If a spring,with spring constant α, is stretched by an extension x, *Hooke's Law* states that the force on the particle has magnitude $\alpha\,|x|$ towards the equilibrium. Thus whether the extension x is positive (an extension) or negative (a compression) Newton's Second Law gives

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\alpha x.$$

Show that the general solution of this equation is

$$x(t) = A\cos\omega t + B\sin\omega t,$$

for constants A and B, and where $\omega^2 = \alpha/m$. Show that

$$\frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}\alpha x^2 = E$$

is constant throughout the motion. What does the quantity $\frac{1}{2}\alpha x^2$ represent?

3. Say that the particle and spring in Question 2 lie on a rough table, so that there is a resistant frictional force of magnitude μmg (coefficient of friction μ) *when the particle is in motion* and we have

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\alpha x + \mu mg \qquad \text{when } x \geqslant 0.$$

Say that we have initially $x(0) = \varepsilon > 0$ and $x'(0) = 0$. What happens if $\mu \geqslant \alpha\varepsilon/(mg)$? Show that if

$$\frac{\varepsilon\alpha}{2mg} < \mu < \frac{\varepsilon\alpha}{mg}$$

then the particle comes to rest before its normal equilibrium position, and find the value of x where this occurs.

Show that

$$\frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}\alpha x^2 = \frac{1}{2}\alpha\varepsilon^2 + \mu mg(x - \varepsilon),$$

throughout the motion and explain the significance of the terms in this identity.

4. Consider a mass m at the end of a light inextensible rod of length l making small swings under gravity; let θ denote the angle the rod makes with the vertical.

(i) Note that $\mathbf{r} = (l\sin\theta, -l\cos\theta)$. What do $\mathrm{d}\mathbf{r}/\mathrm{d}t$ and $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ equal?

(ii) Use Newton's Second Law to show that

$$l\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -g\sin\theta,$$

and find an expression for the tension in the rod.

(iii) Show throughout the motion that $\frac{1}{2}ml^2(\mathrm{d}\theta/\mathrm{d}t)^2 - mgl\cos\theta = E$ is constant.

(iv) Say that the pendulum's oscillations are small enough that the approximation $\sin\theta \approx \theta$ applies. Show that the pendulum's swings have period $2\pi\sqrt{l/g}$.

(v) More generally if the particle starts off with $\theta = \alpha, \mathrm{d}\theta/\mathrm{d}t = 0$, show that the oscillations have exact period

$$4\sqrt{\frac{l}{2g}}\int_0^\alpha \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$