GEOMETRY EXERCISES

1. Describe the regions of space given by the following vector equations. In each, $\mathbf{r}$ denotes the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; '$\cdot$' and $\wedge$ denote the scalar (dot) and vector (cross) product:

- $\mathbf{r} \wedge (\mathbf{i} + \mathbf{j}) = (\mathbf{i} - \mathbf{j})$,
- $\mathbf{r} \cdot \mathbf{i} = 1$,
- $|\mathbf{r} - \mathbf{i}| = |\mathbf{r} - \mathbf{j}|$,
- $|\mathbf{r} - \mathbf{i}| = 1$,
- $\mathbf{r} \cdot \mathbf{i} = \mathbf{r} \cdot \mathbf{j} = \mathbf{r} \cdot \mathbf{k}$,
- $\mathbf{r} \wedge \mathbf{i} = \mathbf{i}$.

2. Find the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z}{2} \quad \text{and} \quad x = 2, \quad \frac{y-1}{2} = z.$$

[Hint: parametrise the lines and write down the vector between two arbitrary points on the lines; then determine when this vector is normal to both lines.]

3. Let L_θ denote the line through (a, b) making an angle θ with the x-axis. Show that L_θ is a tangent of the parabola $y = x^2$ if and only if

$$\tan^2\theta - 4a\tan\theta + 4b = 0.$$

[Hint: parametrise L_θ as $x = a + \lambda\cos\theta$ and $y = b + \lambda\sin\theta$ and determine when L_θ meets the parabola precisely once.]

Show that the tangents from (a, b) to the parabola subtend an angle $\pi/4$ if and only if

$$1 + 24b + 16b^2 = 16a^2.$$

[Hint: use the formula $\tan(\theta_1 - \theta_2) = (\tan\theta_1 - \tan\theta_2)/(1 + \tan\theta_1 \tan\theta_2)$.]

Sketch the curve $1 + 24y + 16y^2 = 16x^2$ and the original parabola on the same axes.

4. What transformations of the xy-plane do the following matrices represent:

$$i) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad ii) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$iii) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad iv) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Which, if any, of these transformations are invertible?

5. The *cycloid* is the curve given parametrically by the equations

$$x(t) = t - \sin t, \quad \text{and} \quad y(t) = 1 - \cos t \quad \text{for } 0 \le t \le 2\pi.$$

(a) Sketch the cycloid.

(b) Find the arc-length of the cycloid.

(c) Find the area bounded by the cycloid and the x-axis.

(d) Find the area of the surface of revolution generated by rotating the cycloid around the x-axis.

(e) Find the volume enclosed by the surface of revolution generated by rotating the cycloid around the x-axis.