Future stability of models of the universe

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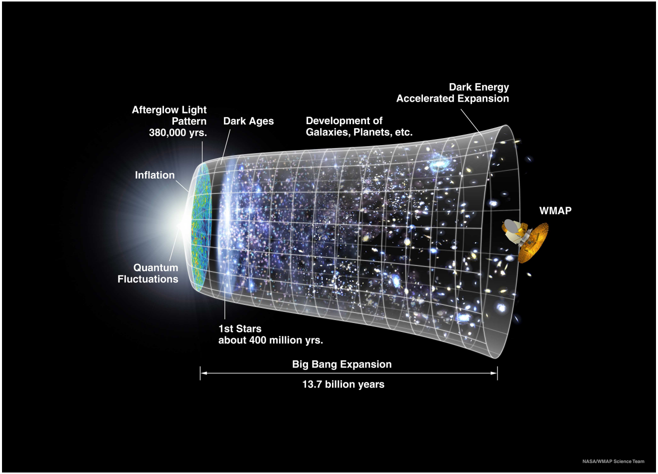
Introduction

The standard models of the universe

- satisfy the **cosmological principle** (i.e., they are spatially homogeneous and isotropic),
- are spatially flat,
- have matter content consisting of ordinary matter, dark matter and dark energy.

Current model of the universe



Current model of the universe: NASA/WMAP Science Team.

Questions

Since the universe is not exactly spatially homogeneous and isotropic, it is natural to ask: **are the standard models future stable?**

Since the assumption of spatial homogeneity and isotropy yields strong restrictions on the allowed topologies, it is also of interest to ask: **what are the restrictions on the global topology of the universe imposed by the constraint that what we observe seems to be close to a standard model?**

Previous results

- Stability of de Sitter space in $3+1$-dimensions, etc., Helmut Friedrich, '86, '91.
- Stability of even dimensional de Sitter spaces, Michael Anderson '05.
- Stability in the non-linear scalar field case, H.R. '08.
- Einstein-Euler with a positive cosmological constant, Igor Rodnianski and Jared Speck, *preprint* '09.

Einstein's equations

Einstein's equations:

$$G + \Lambda g = T,$$

where

- (M, g) is a Lorentz manifold,
- $G = \mathrm{Ric} - \frac{1}{2}Sg$ is the Einstein tensor,
- Λ is the cosmological constant,
- T is the stress energy tensor.

Vlasov matter, mathematical structures

In the Vlasov setting, the relevant mathematical structures are

- the **mass shell** P; the future directed unit timelike vectors in (M, g),
- the **distribution function** $f : P \to [0, \infty)$,
- the **stress energy tensor**

$$T_{\alpha\beta}|_{\xi} = \int_{P_{\xi}} f p_{\alpha} p_{\beta} \mu_{P_{\xi}},$$

- the **Vlasov equation**

$$\mathcal{L} f = 0.$$

The Einstein-Vlasov system

The Einstein-Vlasov system consists of the equations

$$\begin{aligned} G + \Lambda g &= T, \\ \mathcal{L}f &= 0 \end{aligned}$$

for g and f. Note that the second equation corresponds to the requirement that f be constant along timelike geodesics.

Induced initial data

Let (M, g, f) be a solution and Σ be a spacelike hypersurface in (M, g). Then the **initial data induced on** Σ consist of

- the Riemannian metric induced on Σ by g, say $\bar{g}$,
- the second fundamental form induced on Σ by g, say $\bar{k}$,
- the induced distribution function $\bar{f} : T\Sigma \to [0, \infty)$.

Here

$$\bar{f} = f \circ \mathrm{proj}_{\Sigma}^{-1},$$

where $\mathrm{proj}_{\Sigma} : P_{\Sigma} \to T\Sigma$ represents projection orthogonal to the normal.

Constraint equations

Due to the fact that (M, g, f) solve the Einstein-Vlasov system, $(\bar{g}, \bar{k}, \bar{f})$ have to solve the **constraint equations**:

$$\begin{aligned} \bar{r} - \bar{k}_{ij}\bar{k}^{ij} + (\mathrm{tr}\bar{k})^2 &= 2\Lambda + 2\rho^{\mathrm{Vl}}, & (1) \\ \overline{\nabla}^j \bar{k}_{ji} - \overline{\nabla}_i(\mathrm{tr}\bar{k}) &= -\bar{J}_i^{\mathrm{Vl}}. & (2) \end{aligned}$$

Here

$$\begin{aligned} \rho^{\mathrm{Vl}}(\xi) &= \int_{T_\xi\Sigma} \bar{f}(\bar{p})[1 + \bar{g}(\bar{p}, \bar{p})]^{1/2} \bar{\mu}_{\bar{g},\xi}, & (3) \\ \bar{J}^{\mathrm{Vl}}(\bar{X}) &= \int_{T_\xi\Sigma} \bar{f}(\bar{p})\bar{g}(\bar{X}, \bar{p})\bar{\mu}_{\bar{g},\xi}. & (4) \end{aligned}$$

Abstract initial data

Abstract initial data consist of

- an n-dimensional manifold Σ,
- a Riemannian metric $\bar{g}$ on Σ,
- a symmetric covariant 2-tensor field $\bar{k}$ on Σ,
- a function $\bar{f} : T\Sigma \to [0, \infty)$ (belonging to a suitable function space),

all assumed to be smooth and such that the constraint equations (1) and (2) are satisified.

Developments

Given abstract initial data, a **development** is

- a solution (M, g, f) to the Einstein-Vlasov system, and
- an embedding $i : \Sigma \to M$

such that the initial data induced on $i(\Sigma)$ by (M, g, f) correspond to the abstract initial data.

Function spaces

If Σ is a compact manifold, $\bar{\mathfrak{D}}^{\infty}_{\mu}(T\Sigma)$ denotes the space of smooth functions $f : T\Sigma \to \mathbb{R}$ such that

$$\|\bar{f}\|_{H^{l}_{\mathrm{Vl},\mu}} = \left(\sum_{i=1}^{j} \sum_{|\alpha|+|\beta|\leq l} \int_{\bar{\mathrm{x}}_i(U_i)\times\mathbb{R}^n} \langle\bar{\varrho}\rangle^{2\mu+2|\beta|} \bar{\chi}_i(\bar{\xi}) (\partial^{\alpha}_{\bar{\xi}} \partial^{\beta}_{\bar{\varrho}} \bar{\mathrm{f}}_{\bar{\mathrm{x}}_i})^2 (\bar{\xi}, \bar{\varrho}) d\bar{\xi} d\bar{\varrho} \right)^{1/2}$$

is finite for every $l \geq 0$.

Bianchi initial data

Let

- G be a 3–dimensional Lie group,
- $5/2 < \mu \in \mathbb{R}$,
- $\bar{g}$ and $\bar{k}$ be a left invariant Riemannian metric and symmetric covariant 2–tensor field on G respectively,
- $\bar{f} \in \bar{\mathfrak{D}}^{\infty}_{\mu}(TG)$ be left invariant and non-negative.

Then $(G, \bar{g}, \bar{k}, \bar{f})$ are referred to as **Bianchi initial data** for the Einstein–Vlasov system, assuming they satisfy the constraints.

Future stability of spatially locally homogeneous solutions

Let $(G, \bar{g}_{\mathrm{bg}}, \bar{k}_{\mathrm{bg}}, \bar{f}_{\mathrm{bg}})$ be Bianchi initial data for the Einstein–Vlasov system, where

- the universal covering group of G is not isomorphic to $\mathrm{SU}(2)$,
- $\mathrm{tr}\,\bar{k}_{\mathrm{bg}} = \bar{g}^{ij}_{\mathrm{bg}}\bar{k}_{\mathrm{bg},ij} > 0$.

Assume that there is a cocompact subgroup Γ of the isometry group of the initial data. Let Σ be the compact quotient. Then there is an $\epsilon > 0$ such that if $(\Sigma, \bar{g}, \bar{k}, \bar{f})$ are initial data satisfying

$$\|\bar{g} - \bar{g}_{\mathrm{bg}}\|_{H^5} + \|\bar{k} - \bar{k}_{\mathrm{bg}}\|_{H^4} + \|\bar{f} - \bar{f}_{\mathrm{bg}}\|_{H^4_{\mathrm{Vl},\mu}} \leq \epsilon,$$

then the maximal Cauchy development of $(\Sigma, \bar{g}, \bar{k}, \bar{f})$ is future causally geodesically complete.

Question, topology

Assume that

- the observational data indicate that, to our past, the universe is well approximated by one of the standard models,
- interpreting the data in this model, we only have information concerning the universe for $t \geq t_0$,
- there is a big bang,
- analogous statements apply to all observers in the universe (with the same t_0).

The question is then: what conclusions are we allowed to draw concerning the global spatial topology of the universe?

Ingredients

Assume we are given

- a standard model, characterised by an existence interval I, a scale factor a etc.,
- a $t_0 \in I$, which represents the time to the future of which we wish the approximation to be valid,
- an $l \in \mathbb{N}$, specifying the norm with respect to which we measure proximity to the standard model,
- an $\epsilon > 0$, characterising the size of the distance,
- a closed 3-manifold Σ.

Construction

There is a solution (M, g, f) with the following properties:

- (M, g, f) is a maximal Cauchy development,
- (M, g) is future causally geodesically complete,
- there is a Cauchy hypersurface, say $\bar{S}$, in (M, g), diffeomorphic to Σ,
- given an observer γ in (M, g), there is a neighbourhood, say U, of
$$J^-(\gamma) \cap J^+(\bar{S})$$
such that the solution in U is ϵ-close to the standard model in a solid cylinder of the form $[t_0, \infty) \times \bar{B}_R(0)$,
- all timelike geodesics in (M, g) are past incomplete,
- the solution is stable with these properties.