M.Sc. in Mathematical Modelling and Numerical Analysis

Paper A (Mathematical Modelling)

Thursday 21 April, 1994, 9.30 a.m. – 12.30 p.m.

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1. (i) Starting from Fourier's theorem stating that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

where a_n and b_n are the Fourier coefficients as usually defined, give a heuristic (simple-minded) argument to show that if

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x$$

then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(k)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k.$$

State any necessary conditions that should be imposed on f.

(ii) Obtain Parseval's identity in the form

$$\langle \widehat{f}, g\rangle = \langle f, \widehat{g}\rangle,\ \text{where } \langle F, G\rangle := \int_{-\infty}^{\infty} F(t)G(t)\mathrm{d}t.$$

(iii) Obtain the Fourier transforms of $\dfrac{\mathrm{d}f}{\mathrm{d}x}$, xf and $\displaystyle\int_{-\infty}^{\infty} f(s)g(x-s)\mathrm{d}s$ in terms of $\widehat{f}$ and $\widehat{g}$.

(iv) List some criteria that a boundary-value problem for a partial differential equation should satisfy if it is to be susceptible to the application of a Fourier transform.

2. (i) Explain what is meant by saying that a function $f(x)$ has an *asymptotic expansion* as $x \to \infty$. Show that, when x is real,

$$\mathrm{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2} \mathrm{d}t$$

has the expansion

$$\mathrm{erfc}(x) \sim \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi} x} \left(1 + O\left(\frac{1}{x^2}\right)\right)$$

as $x \to +\infty$. Over what region of complex x would you expect this formula to apply? Can you find a formula that is valid as $x \to -\infty$?

(ii) Explain how the method of stationary phase can be used to compute the asymptotic expansion of

$$\int_a^b \mathrm{e}^{\mathrm{i}xf(k)} g(k) \mathrm{d}k, \quad f, g \text{ real},$$

as $x \to +\infty$ in the case where f has a vanishing derivative at one point within the range of integration.

A wave is described by

$$\int_{-\infty}^\infty a(k) \mathrm{e}^{\mathrm{i}k(x - c(k)t)} \mathrm{d}k.$$

Evaluate this integral as $x, t \to +\infty$ with $V := x/t = O(1)$, and show that the amplitude is greatest when

$$V = \frac{\mathrm{d}}{\mathrm{d}k}(kc(k)).$$

3. What is meant by the statement that

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial y} = \mathbf{c}$$

is *quasilinear*? Consider the Cauchy problem for a quasilinear system with $\mathbf{u} = \mathbf{u}_0(s)$ on $x = x_0(s)$, $y = y_0(s)$ as prescribed data:

(i) Show that the derivatives of $\mathbf{u}$ on $x = x_0$, $y = y_0$ can be calculated formally as long as $\det\left(\mathbf{B} - \frac{y_0'}{x_0'}\mathbf{A}\right) \neq 0$.

(ii) Define *well-posedness* and state conditions under which the Cauchy problem is well posed.

(iii) In the case that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{u}_0$, x_0, y_0 are analytic functions of all their arguments, what does the Cauchy–Kowalewski theorem say?

(iv) In the case that a quasilinear system is in conservation form

$$\frac{\partial}{\partial x}(\mathbf{A}\mathbf{u}) + \frac{\partial}{\partial y}(\mathbf{B}\mathbf{u}) = \mathbf{0}, \quad -\infty < x < \infty,\ y > 0,$$

with $x_0(s) = s$, $y_0(s) = 0$, explain what is meant by a *weak solution*.

Consider the case

$$\frac{\partial u}{\partial y} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0;$$

show that

$$u = \begin{cases} 0 & x < y/2 \\ 1 & x > y/2 \end{cases} \text{ and } u = \begin{cases} 1 & x < y/2 \\ 0 & x > y/2 \end{cases}$$

are both weak solutions of this conservation law but that only one of them satisfies causality as y increases.

4. Show that the solution of the linear second-order ordinary differential equation

$$\mathcal{L}u(x) = f(x), \quad 0 < x < 1,$$

with $u(0) = u(1) = 0$, can be written as

$$u(\xi) = \int_0^1 G(x, \xi) f(x) \mathrm{d}x$$

where G (as a function of x) satisfies a differential equation and boundary conditions that you should define.

Show further that if $\mathcal{L}u(x) = f(x)$, $x > 0$, with $u(0) = u'(0) = 0$, then

$$u(\xi) = \int_0^\xi R(x, \xi) f(x) \mathrm{d}x$$

where R also satisfies a differential equation and boundary conditions that you should define.

Generalise these formulae to

(i) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$, $(x, y) \in \Omega$, with $u = 0$ on $\partial\Omega$, for a finite region Ω, and

(ii) $\dfrac{\partial^2 u}{\partial x \partial y} = f(x, y)$ with $u = \dfrac{\partial u}{\partial n} = 0$ on an open curve Γ nowhere parallel to the axes.

In each case write u as a double integral over a region that you should specify and indicate the nature of the singularities of G and R, respectively.

Show in (i) that if Ω is $y > 0$ and suitable assumptions are made about the behaviour of u and f at infinity then

$$G = \frac{1}{4\pi} \log \frac{(x - \xi)^2 + (y - \eta)^2}{(x - \xi)^2 + (y + \eta)^2}.$$

What is R in (ii)?

5. (i) Show that the flow of an incompressible inviscid fluid can be modelled by the equations

$$\nabla \cdot \mathbf{u} = 0$$
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p$$

where $\mathbf{u}$ is the velocity, ρ the density and p the pressure and where body forces are neglected. Cite typical initial and boundary conditions for $\mathbf{u}$ and p. Show that in the case of irrotational flow

$$p + \frac{1}{2}\rho\,|\mathbf{u}|^2 + \frac{\partial \phi}{\partial t} = 0$$

where $\mathbf{u} = \nabla\phi$. How does this relationship between velocity and pressure differ from that which exists in porous medium flow?

(ii) Show that heat transfer in a heat-conducting material can be modelled by

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T$$

where ρ is the density, c the specific heat, k the conductivity (all constant) and T is the temperature.

Suppose that a phase change occurs across the surface $f(x, y, z, t) = 0$ with a release of latent heat L per unit mass. Show that

$$[k\nabla T \cdot \nabla f]_S^L = \rho L \frac{\partial f}{\partial t}$$

where $[\;]_S^L$ denotes the jump from solid to liquid.

What second boundary condition could be applied on $f = 0$?

6. The Chezy relation between mean velocity and channel depth in river flow is

$$u = C(RS)^{1/2},$$

where R is the hydraulic depth and S is the slope; R is defined as A/l, where l is the cross-sectional perimeter and A is the cross-sectional area. Assuming that l is constant, derive a model equation for A and show that, by suitably non-dimensionalising it, it can be written in the form

$$\frac{\partial A}{\partial t} + A^m \frac{\partial A}{\partial s} = 0,$$

where m should be specified. Deduce that wave-like disturbances can propagate downstream, and give their speed in terms of the mean velocity.

A river is supplied by groundwater flow at a constant rate q (volume per unit length of river per unit time). Derive the corresponding equation for A and find its steady solution.

Measurements indicate that $l = cQ^{1/3}$, where Q is the river discharge (volume flux). Describe how this affects the steady states and wave speeds of the model.