M.Sc. in Mathematical Modelling and Numerical Analysis

Paper A (Mathematical Modelling)

Thursday 20 April, 1995, 9.30 a.m. – 12.30 p.m.

All candidates are expected to attempt at least three parts of question 1. Having done so, you may attempt as many of the remaining questions (including the remaining part of question 1) as you wish.
The separate parts of questions 2-5 can be attempted independently.

1. (i) The equations of plane strain in linear elasticity, with displacements $(u, 0, w)$, can be written as

$$\tau = \frac{E}{(1+\nu)}\varepsilon_{13}, \tag{1a}$$

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_1 + \nu\varepsilon_3], \tag{1b}$$

$$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_1 + (1-\nu)\varepsilon_3], \tag{1c}$$

$$\varepsilon_1 = \frac{\partial u}{\partial x}, \tag{2a}$$

$$\varepsilon_3 = \frac{\partial w}{\partial z}, \tag{2b}$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{2c}$$

$$\frac{\partial \sigma_1}{\partial x} + \frac{\partial \tau}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{3a}$$

$$\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_3}{\partial z} - \rho g = \rho\frac{\partial^2 w}{\partial t^2}. \tag{3b}$$

What modelling and physical principles are expressed by these equations?

A thin horizontal beam of thickness h and length l undergoes small deformations $(u, 0, w)$ in a vertical plane. By assuming that $\sigma_3, \varepsilon_3, \varepsilon_{13}, \rho\partial^2 u/\partial t^2$ are small in equations (1c), (2b), (2c) and (3a) above, show that the bending moment M and shear force T, given by

$$M = \int_{-h/2}^{h/2} z\sigma_1 dz, \qquad T = \int_{-h/2}^{h/2} \tau dz,$$

satisfy the approximate equations

$$\frac{\partial M}{\partial x} = T, \qquad \frac{\partial T}{\partial x} - \rho g h = \rho h\frac{\partial^2 w}{\partial t^2}.$$

Hence deduce the Euler-Bernoulli beam equation in the form

$$\rho h\frac{\partial^2 w}{\partial t^2} + Dw\frac{\partial^4 w}{\partial x^4} = -\rho g h,$$

where the flexural rigidity $D = Eh^3/12(1-\nu^2)$. (A full non-dimensional derivation is not required.)

QUESTION CONTINUES ON NEXT PAGE

1. (CONTINUED FROM PREVIOUS PAGE)

(ii) A lake of depth h freezes over in winter. The surface $z = 0$ is maintained at a temperature $T = -\Delta T$ °C, while the bottom $z = -h$ is supplied with a geothermal heat flux G. Write down a model for the temperature field in the ice and in the water, assuming that the two phases have equal thermal conductivities, densities and specific heat. By non-dimensionalising the model, show that the solution depends on two parameters $\sigma = L/c_p\Delta T$ and $\gamma = Gh/k\Delta T$, where L is latent heat, c_p is specific heat, k is thermal conductivity.

Show that if $\sigma \gg 1$, then the temperature field is quasi-static and the dimensionless ice-water interface position $z = s(t)$ satisfies the approximate equation

$$\dot{s} \approx -\frac{1}{\sigma}\left(\frac{1}{s} + \gamma\right).$$

(iii) The reactions which are thought to control the Belousov-Zhabotinskii reaction are

$$Br^- + BrO_3^- + 2H^+ \xrightarrow{K_1} HBrO_2 + HOBr,$$

$$Br^- + HBrO_2 + H^+ \xrightarrow{K_2} 2HOBr,$$

$$2Ce^{3+} + BrO_3^- + HBrO_2 + 3H^+ \xrightarrow{K_3} 2Ce^{4+} + 2HBrO_2 + H_2O,$$

$$2HBrO_2 \xrightarrow{K_4} HOBr + BrO_3^- + H^+,$$

$$4Ce^{4+} + BrCH(COOH)_2 + 2H_2O \xrightarrow{K_5} 4Ce^{3+} + HCOOH + 2CO_2 + 5H^+ + Br^-.$$

In the Field–Körös–Noyes model for these reactions, the concentration variables $X = [HBrO_2]$, $Y = [Br^-]$ and $Z = [Ce^{4+}]$ satisfy the ordinary differential equations

$$\begin{aligned}
\dot{X} &= K_1AY - K_2XY + K_3AX - 2K_4X^2,\\
\dot{Y} &= -K_1AY - K_2XY + fK_5Z,\\
\dot{Z} &= 2K_3AX - K_5Z.
\end{aligned}$$

where $A = [BrO_3^-]$.

State the law of mass-action, and explain how it is used to derive these equations from the reaction scheme above. What is the rôle of the coefficient f?

What features of the reaction would you expect to be able to predict using this model?

QUESTION CONTINUES ON NEXT PAGE

1. (CONTINUED FROM PREVIOUS PAGE)

(iv) A dimensionless model for spruce budworm populations in the forests of New Brunswick is

$$\begin{aligned}
\varepsilon_1 \dot{B} &= B\left[1 - \frac{B}{S}\left(\frac{\delta^2 + E^2}{E^2}\right)\right] - \frac{\lambda B^2}{(\nu^2 S^2 + B^2)}, \\
\dot{S} &= S(1 - S/E), \\
\varepsilon_2 \dot{E} &= E(1 - E) - \gamma \frac{B}{S}\left(\frac{E^2}{\delta^2 + E^2}\right).
\end{aligned}$$

Explain briefly the meaning of the variables and of the terms in the equations. Suppose that $\varepsilon_1 \ll 1$, $E \gg \delta$, and that $\lambda \sim \nu \ll 1$. Show that equilibria of the equation for B lead to the equation

$$\left(\frac{\nu S}{\lambda}\right)\left[1 - \nu\left(\frac{B}{\nu S}\right)\right] = \frac{(B/\nu S)}{1 + (B/\nu S)^2}.$$

By graphical means, or otherwise, show that multiple equilibria exist if, approximately, $4\lambda < S < \lambda/2\nu$. Sketch the equilibrium curve in the $S - (B/\nu S)$ plane, and explain briefly how relaxation oscillations can occur.

2. The three parts of this question relate to the differential equations for $f(r)$,

$$f'' + \frac{1}{r}f' \mp f = 0, \qquad 0 < r < \infty. \tag{1}$$

(i) If $\phi(x,y)$ is prescribed on the boundary $\partial\Omega$ of a region $\Omega \subseteq \mathbb{R}^2$, and makes

$$\int\int_\Omega \left(|\nabla\varphi|^2 \pm \varphi^2\right) \mathrm{d}x\mathrm{d}y$$

stationary, show that

$$\nabla^2\varphi \mp \varphi = 0$$

in Ω. When Ω is $\mathbb{R}^2$ and $\varphi = f(r)$, show that f satisfies (1).

(ii) When f satisfies (1) with the plus sign, show that $\bar{f}(p) = \int_0^\infty f(r)\mathrm{e}^{-pr}\mathrm{d}r$ is given by

$$\bar{f}(p) = A/\sqrt{1+p^2}$$

for some constant A. Hence show that, if $f(0) = 1$,

$$f(r) = \frac{2}{\pi}\int_0^1 \frac{\cos rs \ \mathrm{d}s}{\sqrt{1-s^2}}.$$

Why has this approach only generated one solution to a second-order ordinary differential equation?

(iii) Use the WKB method to show that the solutions of (1) with the plus sign satisfy

$$f(r) \sim \text{const.}\frac{\mathrm{e}^{\pm ir}}{\sqrt{r}}\left(1 + O\left(\frac{1}{r}\right)\right)$$

as $r \to \infty$. What happens with the minus sign?

3. (i) What is meant by the statement that

$$f(x,\epsilon) \sim \sum_{0}^{\infty} \alpha_n(\epsilon) f_n(x)$$

as $\epsilon \to 0$, where $|\alpha_{n+1}/\alpha_n| \to 0$ as $\epsilon \to 0$? Plot a typical graph of $f(x,\epsilon) - \sum_0^N \alpha_n(\epsilon) f_n(x)$ as a function of N for a fixed x and several fixed, small values of ϵ.

(ii) Show that as $x \to +\infty$

$$\int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t \sim \frac{\mathrm{e}^{-x^2}}{2x}\left(1 + O\left(\frac{1}{x^2}\right)\right);$$

what happens as $x \to -\infty$?

(iii) Describe the multiple scales method for finding asymptotic solutions of o.d.e.'s of the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x + \epsilon f(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t) = 0$$

as $\epsilon \to 0$. What phenomena can it predict for

(a) the Van der Pol equation, $f = (1 - x^2)\mathrm{d}x/\mathrm{d}t$;

(b) Duffing's equation, $f = \lambda x - x^3 - \cos t$ (where $\lambda = \text{const.}$)?

[Detailed calculations are NOT required.]

4. (i) Under what conditions on the equation $A\mathbf{x} = \mathbf{0}$ does the matrix equation

$$A\mathbf{x} = \mathbf{b} \quad \text{(where } A \text{ is square)}$$

have a unique solution? When it does not have a unique solution, what conditions must be satisfied for there to be any solution at all?

(ii) Under what conditions on the equation $\mathcal{L}u = 0$, with $u(0) = u(1) = 0$, does the second order linear o.d.e. $\mathcal{L}u = f$, $u(0) = u(1) = 0$, have a unique solution? Can $\mathcal{L}u = f$ with $u(0) = u'(0) = 0$ have a nonunique solution? In each of these two cases, describe the singularity at $x = \xi$, of the relevant Green's function $G(x,\xi)$, where

$$\mathcal{L}G = \delta(x - \xi).$$

(iii) Suppose $\mathcal{L}_{xy}G = \delta(x-\xi)\delta(y-\eta)$. Describe the singularity of $G(x,y;\xi,\eta)$ at $(x,y) = (\xi,\eta)$ in the two cases

(a) $\mathcal{L} = \partial^2/\partial x^2 + \partial^2/\partial y^2$

(b) $\mathcal{L} = \partial^2/\partial x^2 - \partial^2/\partial y^2$.

Find G explicitly in case (a) when $G = 0$ on $x^2 + y^2 = 1$.

5. (i) Show that the Rankine–Hugoniot condition for a weak solution of the conservation law

$$\frac{\partial}{\partial t}\Big(P(u)\Big) + \frac{\partial}{\partial x}\Big(Q(u)\Big) = 0$$

is $\mathrm{d}x/\mathrm{d}t = [Q]/[P]$, where [ ] denotes the jump from one side of a shock to the other. Show that if

$$P = u, \quad Q = \frac{1}{2}u^2, \text{ and } u = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \qquad \text{at } t = 0,$$

then

$$u = \begin{cases} 0, & x < t/2 \\ 1, & x > t/2 \end{cases}$$

is a weak solution. Show further that

$$u = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & t < x \end{cases}$$

is also a weak solution. Which one might satisfy the limit as $\epsilon \to 0$ of

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2} \qquad ?$$

(ii) Show that Charpit's equations for the p.d.e.

$$H\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

imply that $\dot{x} = \partial H/\partial y, \quad \dot{y} = -\partial H/\partial x$.
Show that these equations describe straight rays in the case

$$H = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 1.$$

What are the level curves of u in this case?
Draw the rays in the interior of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

given that $u = 0$ on the boundary.