M.Sc. in Mathematical Modelling and Numerical Analysis

Paper A (Mathematical Modelling)

Thursday 24 April, 1997, 9.30 a.m. – 12.30 p.m.

All candidates are expected to attempt one (but **not** *both) of the Questions 1's, which carry one-third of the marks of the entire paper.*

Either *Option A: Mathematical Modelling*

or *Option B: Financial Modelling.*

Candidates may then attempt as many of questions 2–5 as they wish.

If a candidate is taking either Mathematical Modelling or Finance as a Special Topic, they are not allowed to answer the Question 1 on that topic and so must attempt the other question.

Please answer Question 1 on a separate sheet.

Do not turn this page until you are told that you may do so

1. (Option A: Mathematical Modelling)

You are expected to attempt at least three parts.

(a) A lake of depth h at initial temperature T_0, the freezing point of water, is cooled from above by imposition of a surface temperature $T_s = T_0 - \Delta T$. Write down a model for the one-dimensional temperature profile, and non-dimensionalise the model. Evaluate the Stefan number $St = L/c_p\Delta T$ if the latent heat $L = 3.3 \times 10^5$ J kg^{-1}, the specific heat $c_p = 2 \times 10^3$ J kg^{-1} K^{-1}, and $\Delta T = 10$ K, and use this value to obtain an approximate solution. Hence show that the time to freeze the lake solid is approximately

$$\frac{\rho L h^2}{2k\Delta T},$$

where ρ is the density of ice, and k is its thermal conductivity.

(b) Lava is emitted (slowly) from a fissure, and flows viscously outwards above a horizontal plane. Assuming a two-dimensional model of slow viscous flow under gravity, show that the surface profile $z = \eta(x, t)$ satisfies the approximate equation

$$\frac{\partial \eta}{\partial t} = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left[\eta^3 \frac{\partial \eta}{\partial x}\right],$$

where z is height above the plane and x is the horizontal distance from the fissure. Here ρ is the density, g is gravity and μ is the viscosity. Explain any assumptions you make. Show that if the lava injection rate is Q measured as volume per unit length of fissure per unit time (i.e. the cross-sectional area increases as Qt), then a similarity solution exists in the form $\eta = t^\beta f(\xi)$, $\xi = x/t^\alpha$ and determine the values of α and β, and the equation and boundary conditions satisfied by f. Briefly describe what you expect the solution to look like (*do not* solve the equation).

(c) The temperature of a reacting pellet satisfies the nonlinear equation

$$T_t = T_{xx} + \lambda T^\alpha, \quad -1 < x < 1,$$

for $\lambda > 0$ with $T = 0$ at $x = \pm 1$. Show that the maximum steady state temperature T_m at $x = 0$ satisfies $\lambda = cT_m^{1-\alpha}$, where c should be determined in terms of a definite integral. Sketch T_m as a function of λ for $\alpha > 1$ and $\alpha < 1$, commenting on any differences in stability you might expect in the two cases.

If the boundary condition is modified to $T = T_0 > 0$ at $x = \pm 1$, show that

$$\lambda = T_m^{1-\alpha} c(T_m/T_0),$$

and by describing the behaviour of $c(T_m/T_0)$ for $\alpha > 1$, show that thermal runaway can be expected to occur if λ is large enough.

QUESTION CONTINUES ON NEXT PAGE

1. (CONTINUED FROM PREVIOUS PAGE)

(d) Write down a simple one-dimensional model for a bubbly two-phase flow in a tube $0 < z < l$, for the variables α (void fraction), u (liquid velocity), v (gas velocity) and pressure p (assumed equal between the phases). Your model should include an interactive drag term M (force per unit volume) which opposes the gas motion (but accelerates the liquid), due to the relative motion between the phases. Gravity may be neglected. Give reasons why it is reasonable to neglect the gas acceleration terms if $\rho_g \ll \rho_l$, where these represent the gas and liquid densities, and in this case show that the model reduces to

$$\rho_l(u_t + uu_z) = 2M,$$

$$\alpha_t = [(1-\alpha)u]_z.$$

If $M = \mu D(v-u)/a^2$, where μ is viscosity, a is bubble radius, and D is a drag coefficient, show that $M = \mu D(V-u)/a^2\alpha$, where V is the total volume flux.

Air is admitted with a volume flux V and void fraction α_0 at the inlet $z = 0$, where the liquid velocity $u = 0$. Show that in a steady state, u rapidly adjusts to equal the gas velocity over a distance Δz given by

$$\Delta z \approx \frac{\rho_l V a^2}{\mu D}.$$

Show also that $\alpha \to 1$ in this distance (i.e. the flow turns to a foam).

1. (Option B: Financial Modelling)

You must answer part (a), and at least two others.

(a) In the standard Black-Scholes framework, with constant volatility σ and interest rate r, a European-style derivative on an asset with price S has price $V(S,t)$. Derive the Black-Scholes equation for $V(S,t)$. Comment on the assumptions you have made, and on the role of Δ, the number of assets held in the hedged portfolio. Show how to hedge the Gamma ($\partial^2 V/\partial S^2$) risk in V by hedging with second derivative with price $V_1(S,t)$ as well as the asset.

(b) Suppose the derivative is now an American put option, with strike price E. Explain why its price is different from that of the corresponding European put. Describe in some detail a model and a problem from which it can be found, commenting briefly on how to solve this.

(c) The derivative of part (a) is delta-hedged only at small discrete time intervals δt, and if N of the asset are traded at a price S, a transaction cost $k|N|S$ is incurred. Show that $V(S,t)$ approximately satisfies

$$\mathcal{L}_{BS}V = k\sigma S^2 \sqrt{\frac{2}{\pi \delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right|$$

where $\mathcal{L}_{BS}$ is the Black-Scholes differential operator. Sketch and label on the same graph the values of long and short positions in a European call option, and the Black-Scholes value ($k = 0$).

(d) A derivative pays out \$1 if before time T the asset first rises to an upper value S^+ then falls to a lower value $S^- < S^+$; otherwise it pays nothing. Explain how it can be valued by solving appropriate boundary value problems (which you should specify with reasons, for example in an S–t diagram) for the Black-Scholes equation.

2. (i) State the *Fredholm Alternative* for the problem $\mathcal{L}u = f$ where $\mathcal{L}$ is a linear operator, u is an unknown and f is prescribed. If u and $\mathcal{L}u$ live in a real space on which inner products are defined, what does it mean to say $\mathcal{L}$ is self-adjoint? What does the above say about the existence and uniqueness of the solution of the differential equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = 0$$

on an interval $0 < x < 1$ where $p(x), q(x) > 0$ are continuously differentiable functions on $0 \leq x \leq 1$, subject to the boundary conditions:

(a) $u(0) = a, u(1) = b$,

or

(b) $\frac{du}{dx}(0) = a,\ \frac{du}{dx}(1) = b.$

(ii) Suppose $\mathcal{L}$ in (i) is a real self-adjoint operator and $u(x)$ is a scalar function and that $\mathcal{L}u = \lambda u$ has a discrete spectrum $\lambda = \lambda_i$ for $i = 1, 2, \ldots$. Show that the λ_i are real and that the corresponding eigenfunctions ϕ_i are orthogonal.

Assuming that ϕ_i are orthonormal and complete, show that if the Greens function G is defined to be the solution of $\mathcal{L}G(x, \xi) = \delta(x - \xi)$ then

$$G(x, \xi) = \sum_n \phi_n(\xi)\phi_n(x)/\lambda_n.$$

(iii) Find the complex Fourier transform of

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

in the upper half-plane of the transform variable k. What happens when k is real?

3. (i) What is meant by saying that $\sum_{0}^{\infty} F_n(\varepsilon)$ is an *asymptotic expansion* of a function $F(\varepsilon)$ as $\varepsilon \to 0$?

Suppose the asymptotic solution of

$$\varepsilon^2 y'' = f(x, y)$$

is $y \sim y_0(x) + ...$ outside a boundary layer near $x = 0$ and $y \sim Y_0(X)$ in the boundary layer $X = \frac{x}{\varepsilon} = O(1)$. What does the matching principle tell you about y_0 and Y_0? Use this principle to obtain the first terms in the solution of

$$\varepsilon y'' = (x - 2)y', \quad -1 < x < 1$$

with $y(-1) = 1$, $y(1) = 0$, explaining why the boundary layer is at $x = -1$, and not at $x = 1$.

(ii) Suppose the WKB expansion of the solution of $\varepsilon^2 y'' + xy = 0$ is written as

$$y \sim \exp\{y_0(x)/\varepsilon + y_1(x) + ...\}.$$

Show that $y_0 = \pm\frac{2i}{3}x^{3/2}$ and $\exp(y_1) = cx^{-\frac{1}{4}}$, where c=constant.

(iii) Suppose the differential equation $y'' = f(x, y, y')$ is invariant under the transformation $y \to \tilde{y}$, $x \to \tilde{x} + \lambda$. Reduce it to a first-order equation. What is the property of the transformation that allows this to happen?

Use a similar idea to reduce the linear equation $y'' + f(x)y = 0$ to a first-order equation.

4. Consider the equation

$$A\frac{\partial \mathbf{u}}{\partial x} + B\frac{\partial \mathbf{u}}{\partial y} = \mathbf{c} \qquad (*)$$

where $\mathbf{u}$ is an n-dimensional vector, A, B are $n \times n$ matrices depending on x and y and $\mathbf{c}$ is a vector depending on x, y and $\mathbf{u}$.

(i) If the solution $\mathbf{u}$ satisfies $\mathbf{u} = \mathbf{u}_0(s)$ on the curve Γ given by $x = x_0(s)$, $y = y_0(s)$, show that subject to sufficient conditions on $A, B, \mathbf{c}, x_0, y_0, \mathbf{u}_0$, there is a solution in some neighbourhood of Γ provided $|Ay_0'(s) - Bx_0'(s)| \neq 0$ on Γ. Is this solution unique?

(ii) Show that if (*) is a *hyperbolic* system so that there exist characteristic curves given by $\frac{dy}{dx} = \lambda$ where $|A\lambda - B| = 0$, then along these curves the equation reduces to an ordinary differential equation of the form $\frac{d\mathbf{u}}{dx} = \mathbf{f}$, provided A is a non singular matrix.

(iii) Solve the equations

$$u_x + v_y = 1, \quad v_x + u_y = 1 \qquad (**)$$

subject to $u = 1$, $v = 0$ on $y = \alpha x$ when $\alpha \neq \pm 1$. What can you say about the solution when $\alpha = \pm 1$?

(iv) A solution of the system (**) with Cauchy data $u = 1$, $v = 0$ on $y = 0$ for $x > 0$ and $u = 0$, $v = 0$ on $y = 0$ for $x < 0$ is given by

$$\begin{array}{lll} u = 1 + y, \ \ v = y & \text{for} & y < x \\ u = \frac{1}{2} + y, \ \ v = y - \frac{1}{2} & \text{for} & -y < x < y \\ u = v = y & \text{for} & x < -y. \end{array}$$

Explain carefully what is meant by a *weak solution* and confirm that this is a weak solution of the equations.

5. Derive the conditions satisfied by the Greens Function G which solves the problem

$$\nabla^2\phi = f \quad \text{in} \quad S,$$

$$\phi = g \quad \text{on} \quad \partial S,$$

in the form

$$\phi(\xi,\eta) = \int\int_S G f dS + \int_{\partial S} g \frac{\partial G}{\partial n} ds.$$

[Here ∇^2 is the two dimensional operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and S is a finite region bounded by the closed curve ∂S.]

Show that if G exists, it is unique. If S is the positive quadrant $x > 0$, $y > 0$, what would be appropriate boundary conditions at infinity to make sure that ϕ and G are uniquely determined? Show that in this case

$$G = \frac{1}{4\pi} \ln\left(\frac{((x-\xi)^2 + (y-\eta)^2) \;\; ((x+\xi)^2 + (y+\eta)^2)}{((x-\xi)^2 + (y+\eta)^2) \;\; ((x+\xi)^2 + (y-\eta)^2)}\right),$$

and hence or otherwise show that the solution of

$$\nabla^2\phi = 0 \quad \text{in} \quad y > 0,$$

$$\phi = g(x) \quad \text{on} \quad y = 0, \qquad \phi \to 0 \quad \text{as} \quad y \to \infty,$$

where $g(x)$ is an *odd* function is given by

$$\phi(\xi,\eta) = \frac{\eta}{\pi}\int_0^\infty g(x)\left[\frac{1}{(x-\xi)^2+\eta^2} - \frac{1}{(x+\xi)^2+\eta^2}\right] dx.$$

If the boundary condition $\phi = g$ is replaced by $\frac{\partial \phi}{\partial n} = h$ on ∂S, show that a solution can exist only if

$$\int\int_S f dS = \int_{\partial S} h ds.$$

If this condition is satisfied, will the solution to the problem be unique? By considering the homogeneous problems when ϕ is given or $\frac{\partial \phi}{\partial n}$ is given on ∂S, explain why the existence and uniqueness properties are so different.