M.Phil. in Mathematics for Industry
M.Sc. in Mathematical Modelling and Numerical Analysis

Paper A (Mathematical Modelling)

Thursday 23 April, 1998, 9.30 a.m. – 12.30 p.m.

Candidates must attempt at least one of the questions from option A (i.e. questions 1 and 2 or option B (i.e. questions 3 and 4). They may not attempt questions from both option A and option B.
Candidates may then attempt as many of questions 3–6 as they wish. All questions will carry equal marks.
Please answer option A on a separate sheet.

Do not turn this page until you are told that you may do so

Option A

1. (a) In a single step of the binomial model, the asset price is S_0 at time t, and at time $t + \delta t$ it is $S_u > S_0$ with probability p and $S_d < S_0$ with probability $1 - p$. A derivative contingent on this asset has corresponding prices V_0, V_u, V_d. The continuously compounded risk-free rate is r and no dividends are paid. Find an arbitrage-free formula for V_0 in terms of V_u and V_d. Explain how this approach can be used to calculate the value at time $t = 0$ of a derivative with payoff $V(S,T)$ at time $t = T$.

(b) In continuous time, the evolution of an asset price is modelled by the stochastic differential equation

$$\frac{dS}{S} = (\mu - D)dt + \sigma dX.$$

Explain the meaning of the terms in this equation, and why it is a reasonable model for equity prices.

If $\sigma = 0.2$, $\mu = 0.1$, $D = 0.03$ in annualised units, roughly what proportion of a typical day's price change is attributable to the first term, and what proportion to the second?

(c) Show that, as $x \to -\infty$,

$$\int_{-\infty}^{x} e^{-s^2/2} ds \sim \frac{e^{-x^2/2}}{|x|}(1 + O(1/x)).$$

Use this result to estimate the probability of a fall of 15% or more in a single day of trading when σ is as above. Comment on the result.

2. An asset evolves by the stochastic differential equation of Question 1(b). A derivative contingent on this asset can be shown to satisfy the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0. \tag{1}$$

(a) Find the solution of (1) in which $V = F(t)S$, and interpret it financially.

(b) Calculate the stochastic differential equation followed by $\xi = S^n$, $n > 1$. Hence calculate the value of a derivative that has payoff $V(S,T) = S^n$.

(c) Write down, with justification, boundary value problems which enable you to calculate the value of an option that pays \$1 if, before time T, the asset first falls to a lower value B_-, then rises to a higher value $B_+ > B_-$, and otherwise pays nothing.

Option B

3. A glacier of viscosity η and density ρ flows slowly down a plane slope inclined at an angle α to the horizontal. Supposing the motion to be two-dimensional, use the lubrication approximation to derive the evolution equation for the depth h in the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\rho g \sin\alpha}{3\eta}\left\{1 - \cot\alpha \frac{\partial h}{\partial x}\right\} h^3\right] = a,$$

where x is the downslope coordinate, and specify the meaning of the term a.

If $a(x) = a_0(1 - 2x/l)$, show that a non-dimensional form of the equation can be written as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[(1 - \mu h_x)h^3\right] = 1 - 2x,$$

where μ should be given. If $\mu \ll 1$, find an approximate steady state (you may assume $h = 0$ at $x = 0$), and explain how perturbations to this might behave.

4. Two-dimensional flow in a porous medium is described by the dimensionless equations

$$\nabla^2\psi = -RT_x,$$

$$T_t + \psi_z T_x - \psi_x T_z = \nabla^2 T.$$

Explain the meaning of these equations, and give suitable boundary conditions on the base $z = 0$, and the top $z = 1$, given that the temperature is prescribed on each (the lower surface being hotter), and the base is impermeable, but the top is permeable. Hence show that a steady state solution exists in the form

$$T = 1 - z, \ \psi = 0.$$

Show that small perturbations of the form $\psi = Rl\, f(z)e^{ikx+\sigma t}$, $T - 1 + z = Rl\, g(z)e^{ikx+\sigma t}$ exist, and deduce that f and g satisfy the boundary value problem

$$f'' - k^2 f = -ikRg,$$

$$g'' - k^2 g = \sigma g + ikf,$$

with

$$f(0) = f'(1) = g(0) = g(1) = 0.$$

Hence show that when $\sigma = 0$,

$$\frac{\tanh\alpha_+}{\alpha_+} + \frac{\tan\alpha_-}{\alpha_-} = 0,$$

where $\alpha_+ = (k^2 + k\sqrt{R})^{1/2}$, $\alpha_- = (k\sqrt{R} - k^2)^{1/2}$.

Methods I

5. (i) What is meant by saying that a function $f(\varepsilon)$ has an asymptotic expansion $f(\varepsilon) \sim \sum_{n=0}^{\infty} f_n(\varepsilon)$ as $\varepsilon \to 0$?

(ii) Suppose A is a symmetric matrix and has an eigenvalue λ with eigenvector $\mathbf{X}$. Show that the solution of

$$(A - (\lambda + \varepsilon)I)\mathbf{x} = \mathbf{b}$$

is

$$\mathbf{x} \sim \frac{1}{\varepsilon}\mathbf{x}_0 + \mathbf{x}_1 + \ldots$$

as $\varepsilon \to 0$ and find $\mathbf{x}_0$ in terms of $\mathbf{X}$.

(iii) Suppose $\ddot{x} + (1+\varepsilon)x + K\varepsilon^3 x^3 = \cos t$ where $K = O(1)$. Show that there are solutions with period 2π of the form

$$x \sim \frac{1}{\varepsilon}x_0(t) + x_1(t) + \ldots$$

where $x_0 = A\cos(t+\theta)$ as long as

$$\frac{3}{4}KA^3 + A = \pm 1$$

and $\theta = 0$ or π respectively (N.B. $\cos^3 t \equiv \frac{3\cos t}{4} + \frac{\cos 3t}{4}$).
When the method of multiple scales is used to investigate the stability of this solution, you start by writing

$$x \sim \frac{1}{\varepsilon}x_0(t, \tau) + \ldots.$$

Select a suitable slow scale τ and write down equations to determine x_0.

6. A *fundamental matrix* for the system of ordinary differential equations $\dot{\mathbf{x}} = A(t)\mathbf{x}$ is a square matrix whose columns are independent complementary functions.

(i) If Y and Z are both fundamental matrices, show that

$$Y = ZC$$

for some constant matrix C.

(ii) Suppose $A(t+T) \equiv A(t)$ for some constant T, and that Y is a fundamental matrix. Show that there is another constant matrix D such that

$$Y(t+T) = Y(t)D.$$

Show further that, for another constant matrix B, $Y = Pe^{Bt}$, where P has period T and

$$e^{BT} = D.$$

Supposing that B has independent eigenvectors $\mathbf{e}_i$ with eigenvalues λ_i, show that $\mathbf{e}_i$ are eigenvectors of D and hence give conditions on λ_i for $\mathbf{x}$ either to grow or to remain bounded as $t \to \infty$.

(iii) Suppose the square matrix $G(t, \tau)$ is such that $\frac{dG}{dt} = -GA$, with $G = I$ at $t = \tau$ and that $\mathbf{y}$ satisfies $\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{b}(t)$, $\mathbf{y}(0) = \mathbf{0}$.
Show that $\mathbf{y}(\tau) = G(\tau, \tau)^{-1} \int_0^\tau G(t, \tau)\mathbf{b}(t)dt$.

7. The Green's function $G(\mathbf{x};\xi) = G(x, y; \xi, \eta)$ for Poisson's problem in the plane,

$$\Delta\phi = f(\mathbf{x}) \in \Omega$$

with (*)

$$\phi = g(\mathbf{x}) \in \partial\Omega,$$

is the solution of the equation

$$\Delta G = \delta(x-\xi)\delta(y-\eta)$$

with

$$G = 0 \quad \mathbf{x} \in \delta\Omega.$$

Explain how such a function G enables one to write down the solution of the problem (*) as

$$\phi(\xi) = \int_\Omega G(\mathbf{x};\xi) f(\mathbf{x}) d\mathbf{x} + \int_{\partial\Omega} g(\mathbf{x}) \frac{\partial G}{\partial n}(\mathbf{x};\xi) dS,$$

and by integrating over a suitable subset of Ω, or otherwise, find the appropriate form of the singularity of G at $(x, y) = (\xi, \eta)$.

When Ω is a circle of radius a centred at the origin, use the method of images to show that the corresponding Green's function is

$$G(\mathbf{x};\xi) = \frac{1}{4\pi}\left[\log r_0^2 - \log\left(\frac{|\xi|^2}{a^2} r_1^2\right)\right],$$

where

$$r_0^2 = |\mathbf{x}-\xi|^2 \quad \text{and} \quad r_1^2 = \left|\mathbf{x} - \frac{a^2}{|\xi|^2}\xi\right|^2.$$

Denoting $|\mathbf{x}|$ by r, note that

$$\frac{\partial \mathbf{x}}{\partial r} = \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|},$$

and hence show that

$$\left.\frac{\partial G}{\partial n}\right|_{r=0} = \frac{(a^2 - |\xi|^2)}{2\pi a |\mathbf{x}-\xi|^2}.$$

Deduce that if $f \equiv 0$ in (*), then the value of ϕ at the centre of the circle is the average of its values on the circumference.

8. For which functions $p(w)$ is the system

$$w_t - v_x = 0, \quad v_t + p(w)_x = 0 \tag{$*$}$$

hyperbolic?

For such functions $p(w)$, take suitable linear combinations of the equations to show that the Riemann invariants are

$$v \pm \int^{w} \sqrt{-p'(\xi)}\, d\xi.$$

The particular choice $p(w) = kw^{-\gamma}$, with $k > 0$ and $\gamma \geq 1$ gives a common model of homentropic gas dynamics; for the model to be meaningful, w must be strictly positive. Suppose that $v(x,0)$ and $w(x,0)$ are given, with $w(x,0) > 0$. Evaluate the Riemann invariants for this choice of $p(w)$, assuming that $w(x,t) > 0$ for t positive.

Derive an equation which gives weak solutions of (*) (for this particular $p(w)$), and find the Rankine-Hugoniot conditions which must hold across any shock. Deduce that shocks move with speed c, where

$$c^2 = -\frac{[kw^{-\gamma}]}{[w]}.$$

($[f]$ denotes the jump in f over the shock.)