M.Sc. in Mathematical Modelling and Numerical Analysis

Paper A (Mathematical Modelling)

Thursday 22 April, 1999, 9.30 a.m. – 12.30 p.m.

Candidates may attempt as many of questions as they wish. All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Mathematical Methods I

1. What does it mean to say $\mathcal{L}$ is a real linear, self-adjoint, second-order ordinary differential operator, with homogeneous boundary conditions, in $0 < x < 1$? Suppose such an operator has discrete eigenvalues λ_n and corresponding eigenfunctions ϕ_n, with $\int_0^1 |\phi_n|^2 dx = 1$. Show that λ_n is real and $\int_0^1 \phi_n \phi_m dx = 0$, $n \neq m$.

Show further that if $\mathcal{L}G(x,\xi) = \delta(x-\xi)$ and G satisfies the same homogeneous boundary conditions as ϕ_n, then

$$G(x,\xi) = \sum_n \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n},$$

assuming all λ_n are non-zero.

The generalised Fourier expansion of a function $f(x)$ is defined to be $\sum c_n \phi_n(x)$ where

$$\int_0^1 f(x)\psi(x)dx = \sum_n c_n \left(\int_0^1 \phi_n(x)\psi(x)dx \right)$$

for all suitable test functions $\psi(x)$. Show that if $c_n = \phi_n(\xi)$, then $\int_0^1 f(x)\psi(x)dx = \psi(\xi)$. Hence or otherwise, deduce that $\sum_n \phi_n(x)\phi_n(\xi) = \delta(x-\xi)$.

2. (i) A radio circuit is modelled by

$$\frac{d^2x}{dt^2} + \varepsilon(\lambda - x^2)\frac{dx}{dt} + x = 0$$

where ε is fixed, small and positive. Show that, as λ increases through zero, a periodic solution is born in which x is approximately harmonic in t with amplitude $2\sqrt{\lambda}$. Sketch the $(x, \dot{x})$ phase plane for $\lambda > 0$ and $\lambda < 0$.

(ii) An elastic strut under a compressive force P is modelled by

$$\frac{d^2\theta}{dx^2} + P\sin\theta = 0 \quad \text{with} \quad \theta(0) = \theta(1) = 0.$$

Show that the only small amplitude solution is $\theta \equiv 0$ unless P is near $n^2\pi^2$, where n is an integer. When $P = \pi^2 + \varepsilon$, write $\theta \sim \varepsilon^{1/2}\theta_0(x) + \varepsilon^{3/2}\theta_1(x) + \ldots$ and find θ_0. Draw the response diagram of $\max|\theta|$ as a function of ε.

3. (a) How is the phase plane of the system

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y)$$

related to the Poincaré map $x(0) \to x(T)$, $y(0) \to y(T)$ where T is fixed? When $\dot{x} = y$ and $T = 2\pi/\omega$, draw these Poincaré maps for the non-autonomous step equations

$$\text{(i)} \quad \ddot{x} + x = \cos\omega t; \quad \text{(ii)} \quad \ddot{x} - x = \cos\omega t, \quad \omega \neq 1.$$

(b) Show that the ordinary differential equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0$$

can be reduced to the form $\frac{d^2Y}{dx^2} + g(x)Y = 0$ by writing $y = e^{f(x)}Y(x)$ and choosing f suitably. Reduce this equation for $Y(x)$ to a first-order equation by setting $Y = e^Z$. Why is this possible?

Mathematical Methods II

4. The one-dimensional equations of magnetogasdynamics for high conductivity gas can be written in the form:-

$$\begin{aligned}
\rho_t + u\rho_x + \rho u_x &= 0, \\
B_t + uB_x + Bu_x &= 0, \\
\rho u_t + \rho u u_x + p_x + \frac{1}{\mu}BB_x &= 0, \\
p_t + up_x - \frac{\gamma p}{\rho}(\rho_t + u\rho_x) &= 0,
\end{aligned}$$

where ρ, u, p are the density, velocity and pressure in the gas, B is the magnetic field and μ and γ are constants.

Show that the characteristic velocities $\frac{dx}{dt}$ are given by

$$u, u, u \pm \left(a^2 + \frac{B^2}{\rho\mu}\right)^{\frac{1}{2}},$$

where

$$a^2 = \frac{\gamma p}{\rho}.$$

When $\frac{dx}{dt} = u$, show that there are two left-eigenvectors and evaluate the two Riemann invariants.

If we have no magnetic field i.e. $B = 0$, and homentropic flow so $p = k\rho^\gamma$, with k being constant, show that the system reduces to a 2-dimensional system for ρ and u. Show that the Riemann invariants for this system are $u \pm \frac{2a}{\gamma-1}$ on $\frac{dx}{dt} = u \pm a$ respectively.

5. A function $u(x,t)$ satisfies

$$\mathcal{L}u \equiv u_{xx} - u_t = f(x,t) \tag{1}$$

in a region D bounded by the lines

$$t = 0, \quad t = \tau, \quad x = 0, \quad x = R.$$

State the equation and conditions satisfied by the Green's function $G(x,t;\xi,\tau)$ and show that the solution to (1) is given by

$$\begin{aligned} u(\xi,\tau) \quad = \quad & \int_0^R G(x,0;\xi,\tau)u(x,0)dx + \int_0^\tau u(0,t)G_x(0,t;\xi,\tau)dt \\ & - \int_0^\tau u(R,t)G_x(R,t;\xi,\tau)dt - \int\int_D f(x,t)G(x,t;\xi,\tau)dxdt. \end{aligned}$$

Given that the Green's function for $\mathcal{L}$ with $-\infty < x < \infty$ and $t > 0$ is

$$\frac{1}{2\sqrt{\pi(\tau - t)}} \exp\left(-\frac{(x-\xi)^2}{4(\tau - t)} \right),$$

use the method of images to obtain the Green's function in the quarter space $x > 0$, $t > 0$. Hence find the solution of

$$u_{xx} = u_t, \tag{2}$$

which satisfies

$$\begin{aligned} u = 0, \quad & \text{on} \quad t = 0, \quad x > 0 \\ u = 1, \quad & \text{on} \quad x = 0, \quad t > 0 \\ u \to 0 \quad & \text{as} \quad x \to \infty, \quad t > 0, \end{aligned} \tag{3}$$

leaving your answer in terms of an integral. Hence show that $u(x,t)$ only depends on $x/\sqrt{t}$.

6. (i) Derive Charpit's equations for

$$F(x, y, u, p, q) = 0,$$

in the form

$$\dot{x} = \frac{\partial F}{\partial p}, \quad \dot{y} = \frac{\partial F}{\partial q}, \quad \dot{u} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}, \quad \dot{p} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial u}, \quad \dot{q} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial u},$$

where

$$u = u(x, y), \quad p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}.$$

(ii) Consider the case when

$$F(x, y, u, p, q) \equiv p^4 - p^2 - q^2 = 0.$$

Find the ray equations and show that these are straight lines.
Suppose that $u = 0$ on a circle of radius r, so that the boundary conditions are given by

$$x = x_0(s) = r\cos s, \quad y = y_0(s) = r\sin s, \quad u = u_0(s) = 0$$

for $0 \leq s < 2\pi$. Show that the ray slope is given by

$$-\frac{\sin 2s}{2(1 + \sin^2 s)},$$

and that when $r = 0$, these rays all pass through the origin. Hence draw the ray diagram for $r = 0$ and find the region in $x > 0$ where the rays are confined.