M.Sc. in Mathematical Modelling and Numerical Analysis

Paper B (Numerical Analysis)

Friday 19 April, 1996, 9.30 a.m. – 12.30 p.m.

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1. Compare the advantages of the simple explicit method and the corresponding fully implicit method

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = a\frac{\delta_x^2 U_j^{n+1}}{(\Delta x)^2}$$

for the solution of the heat equation $u_t = au_{xx}$. Describe the Thomas algorithm for the solution of the system of linear equations arising from the fully implicit method, and show that this method for solving the linear equations is numerically stable.

Why is the fully implicit method not equally suitable for the solution of the heat equation $u_t = a(u_{xx} + u_{yy})$ in two space dimensions? Explain the practical advantage of the Douglas-Rachford scheme, on a square mesh with $\nu = a\Delta t/(\Delta x)^2$,

$$\begin{aligned} (1 - \nu\delta_x^2)U^* &= (1 + \nu\delta_y^2)U^n \\ (1 - \nu\delta_y^2)U^{n+1} &= U^* - \nu\delta_y^2 U^n, \end{aligned}$$

and use Fourier analysis to show that the scheme is unconditionally stable.

2. Derive finite difference approximations to the elliptic partial differential equation

$$u_{xx} + u_{yy} = f(x, y)$$

in a region D, for use on a uniform rectangular mesh of size $\Delta x, \Delta y$ suitable for (i) a point whose four nearest neighbours are all internal points of D, (ii) a point for which one neighbour lies on a curved part of the boundary of D, on which a Dirichlet condition is given, (iii) a point which lies on the boundary $y = 0$, on which a Neumann boundary condition is given. Give an expression for the truncation error in each case.

Illustrate your answer for the problem

$$u_{xx} + u_{yy} = x + y$$

where D is the region $0 < x < 2,\ 0 < y < \frac{8-4x}{8-3x}$, with the boundary conditions $\frac{\partial u}{\partial y} = 1$ on $y = 0$, and $u = y$ on the rest of the boundary. Use the mesh size $\Delta x = 1,\ \Delta y = \frac{1}{2}$, and construct a system of two equations which will determine the unknown values of u at the mesh points.

3. Write down the general linear multistep method for the numerical solution of the initial value problem $y' = f(x, y),\ y(0) = y_0$ and define the truncation error. Explain how to determine whether the method is (a) *zero-stable*, (b) *absolutely stable*.

Consider the following linear multistep methods

(i) $y_{n+1} - y_n = hf_{n+1}$

(ii) $y_{n+1} - y_n = h(\frac{3}{2}f_n - \frac{1}{2}f_{n-1})$

(iii) $y_{n+1} - y_{n-1} = 2hf_n$

(iv) $y_{n+1} - 2y_n + y_{n-1} = \frac{1}{2}h(f_{n+1} - f_{n-1})$

where $f_i = f(x_i, y_i)$ and $x_i = ih$ for some fixed step h.

Find the leading terms of the truncation error of each method, determine which are zero-stable, and for those that are zero-stable find limits on the step h for which the method is absolutely stable when applied to the equation $y' = \lambda y, \lambda < 0$. Which method would you choose to solve $10^{-6}y' + y = \cosh x,\ y(0) = 1$. Give reasons.

4. (a) What is an orthogonal matrix? Show that the product of two orthogonal matrices is an orthogonal matrix. Given $A \in \mathbb{R}^{m\times n}$ what properties must U, Σ and V have if $A = U\Sigma V^T$ is a Singular Value Decomposition (SVD) of A? What are the singular values? Using the definition of $\|A\|_2$ in terms of the vector norm defined by $\|x\|_2 = \sqrt{x^T x}$, prove that $\|A\|_2$ is the maximum singular value of A. Suppose now that A is a square and invertible matrix; in terms of the singular values of A what value of $\alpha \in \mathbb{R}$ minimises $\|(A^T A)^{-1} - \alpha I\|_2$?

(b) What is the Jacobi iteration method for a linear system of equations $Ax = b$ or equivalently $\sum_{j=1}^{n} a_{ij}x_j = b_i$ for $i = 1, \ldots, n$? What is the iteration matrix, J? Given that the iteration will converge if and only if all eigenvalues λ of J satisfy $|\lambda| < 1$, prove that the Jacobi iteration will converge if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad \text{for each} \quad i = 1, \ldots, n. \tag{1}$$

Show also that the condition (1) is sufficient for convergence of the Gauss-Seidel iteration.

5. A simple drift-diffusion model for the steady motion of electrons in a semi-conductor gives the equation for the electron density $u(x)$,

$$-[a(x)u' + b(x)u]' + c(x)u = R(x) \quad \text{on} \quad 0 < x < 1.$$

Suppose that $a(x) > 0, c(x) \geq 0$ and $u(0) = u_L, u(1) = u_R$. Let $U(x)$ be the piecewise linear Galerkin approximation to $u(x)$ on the mesh $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$.

Write down the defining relations for U; and, by introducing a local co-ordinate system in each interval, show what integrals need to be evaluated to calculate U exactly. Now assume that a, b, c and R are approximated by constants in each interval, indicated by the subscript i in (x_{i-1}, x_i), and derive the approximate Galerkin equation holding at an interior meshpoint.

State the order of accuracy you would obtain for U and U' when $b(x) \equiv 0$ and the remaining integrals are evaluated exactly, giving brief reasons why; how do these arguments break down when $b \neq 0$?

6. A hierarchy of finite element approximations to Poisson's equation is to be constructed on a bounded domain $\Omega \subset \mathbb{R}^2$, which has been covered by a regular triangulation using straight-sided triangles. Derive the affine transformation from local (ξ, η) co-ordinates to global (x, y) co-ordinates that maps the canonical triangle $[(0,0), (1,0), (0,1)]$ into the generic triangle $[(x_1, y_1), (x_2, y_2), (x_3, y_3)]$; obtain an expression for $|\nabla V|^2$ in terms of the local derivatives $\partial V/\partial \xi$ and $\partial V/\partial \eta$.

Suppose now that the piecewise linear expansion for V, parametrised by its values V_1, V_2 and V_3 at the vertices of the triangle, is extended to a full quadratic form by the addition of terms

$$Q_4 N_4(\xi, \eta) + Q_5 N_5(\xi, \eta) + Q_6 N_6(\xi, \eta),$$

where N_4, N_5 and N_6 are the quadratic basis functions associated with the mid-points of the edges of the triangle. Derive expressions for these basis functions and their coefficients Q_4, Q_5, Q_6, showing that the result forms a conforming approximation and indicating how $|\nabla V|^2$ is calculated.