M.Sc. in Mathematical Modelling and Numerical Analysis

Paper B (Numerical Analysis)

Friday 25 April, 1997, 9.30 a.m. – 12.30 p.m.

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1. Describe the θ-method for the numerical solution of the heat equation $u_t = au_{xx}$ on $0 < x < 1, t > 0$, given the values of $u(x,0), 0 \le x \le 1$, and $u(0,t), u(1,t), t > 0$. Explain how the method is implemented, and give in terms of J an estimate of the number of arithmetical operations to calculate the values $U_j^{n+1}, j = 1, 2, \ldots J-1$. Here U_j^n is the numerical approximation to $u(x_j, t_n)$, where $x_j = j\Delta x, t_n = n\Delta t$.

What is meant by the *truncation error* of the method? Explain briefly why the choice $\theta = \frac{1}{2}$ affects the leading terms of the truncation error.

Show that

$$|U_j^n - u(x_j, t_n)| \le n\Delta t\ T$$

where T is an upper bound for the truncation error, provided that $2(1-\theta)a\Delta t \le (\Delta x)^2$. Comment on this restriction on the size of Δt, in relation to the stability of the method.

2. Define the Lax–Wendroff method for the solution of $u_t + au_x = 0$, where a is a positive constant; use Fourier analysis to determine the leading terms in the amplitude and phase errors. [Note that if $q \sim C_1\xi + C_2\xi^2 + C_3\xi^3 + \ldots$ then $\tan^{-1} q \sim C_1\xi + C_2\xi^2 + (C_3 - \frac{1}{3}C_1^3)\xi^3 + \ldots$.]

Explain what is meant by the *practical stability* of a difference scheme for the solution of $u_t + au_x = bu_{xx}$, where a and b are positive constants. Show that for the explicit scheme using the upwind approximation to u_x the condition for practical stability is

$$\left(\frac{a\Delta t}{\Delta x}\right)^2 \le \frac{a\Delta t}{\Delta x} + 2\frac{b\Delta t}{(\Delta x)^2} \le 1.$$

3. Find the order of the truncation errors for the following methods of approximating the boundary value problem $y'' + f(x, y) = 0$ in $[0, 1], y(0) = y(1) = 0$.

(a) $h^{-2}(y_{r+1} - 2y_r + y_{r-1}) + f(x_r, y_r) = 0, \quad r = 1, \cdots, N-1;$

(b) $h^{-2}(y_{r+1} - 2y_r + y_{r-1}) + \frac{1}{12}(f(x_{r+1}, y_{r+1}) + 10f(x_r, y_r) + f(x_{r-1}, y_{r-1})) = 0,$
$r = 1, \cdots, N-1;$

where $x_r = rh$ with $Nh = 1$ and $y_0 = y_N = 0$.

Show that if

$$y_r = \sum_{s=1}^{N-1} g_{rs} v_s, \quad r = 0, \cdots, N,$$

where

$$g_{rs} = \begin{cases} r(N-s)/N, & 0 \le r \le s \le N, \\ (N-r)s/N, & 0 \le s \le r \le N, \end{cases}$$

then

$$y_{r+1} - 2y_r + y_{r-1} = -v_r, \quad r = 1, \cdots, N-1,$$

for arbitrary $v_1 \cdots, v_{N-1}$.

Show also that

$$\sum_{s=1}^{N-1} g_{rs} = \tfrac{1}{2} r(N-r), \quad r = 1, \cdots, N-1.$$

Suppose that $f(x, y)$ has Lipschitz constant L with respect to y. Show that the solution error $e_r = y(x_r) - y_r$ in method (a) satisfies

$$|e_r| \le h^2 \sum_{s=1}^{N-1} g_{rs}(L|e_s| + |\tau_s|), \quad r = 1, \cdots, N-1,$$

where τ_s is truncation error at x_s.

Deduce that, provided $L < 8$,

$$\max_r |e_r| \le (8 - L)^{-1} \max_r |\tau_r|$$

and hence that method (a) is second order.

Find the corresponding result for method (b).

4. (a) What is a Householder matrix $H(w)$? Prove that if $u, v \in \mathbb{R}^n$ satisfy $u^T u = v^T v$, then there exists $w \in \mathbb{R}^n$ such that $H(w)u = v$ and show how this may be used in computing a QR factorisation of a matrix $A \in \mathbb{R}^{m \times n}$.

Let $b \in \mathbb{R}^m$ be given.

If A is square and non singular, show briefly how the QR factorisation may be used in computing the solution x of $Ax = b$. What factorisation is more usually used for the solution of linear equations?

If $A = QR$ is not square with $m > n$ and $Q^T b = \begin{bmatrix} c \\ d \end{bmatrix}$ with $c \in \mathbb{R}^n, \mathrm{d} \in \mathbb{R}^{m-n}$ prove that

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 = \|\mathrm{d}\|_2.$$

(You may assume that the columns of R are linearly independent.) Using the QR factorisation, how could you calculate the minimising vector x?

(b) If $A, M \in \mathbb{R}^{n \times n}$ are non-singular, show that *if* the iteration

$$Mx^{(k)} = (M - A)x^{(k-1)} + b, \qquad k = 1, 2, \ldots$$

converges for some starting value $x^{(0)}$, then it converges to x^* which satisfies $Ax^* = b$. If $M^{-1}A$ is diagonalisable, establish a condition on the eigenvalues of $M^{-1}A$ which guarantees convergence for any $x^{(0)}$. What governs the rate of convergence? Show how a second sequence of vectors $\{y^{(k)}\}$ can be constructed from the sequence $\{x^{(k)}\}$ so that

$$x^* - y^{(k)} = p_k(T)(x^* - x^{(0)})$$

where p_k is a real polynomial of degree k which satisfies $p_k(1) = 1$ and $T = I - M^{-1}A$. If $T = T^T$, what is $\|p_k(T)\|_2$ and how should the polynomials $\{p_k\}$ ideally be chosen so that $\|x^* - y^{(k)}\|_2$ reduces most rapidly?

5. Solutions of the form $w(x,t) = e^{i\omega t}u(x)$, with boundary conditions $u(0) = 0 = u'(1)$, are sought for the motion of a vibrating system described by

$$w_{tt} = (pw_x)_x - qw, \quad 0 < x < 1, \quad t > 0,$$

where $p(x) \geq p_0 > 0$ and $q(x) \geq 0$. Derive the weak form of the eigenvalue problem for the vibration frequencies $\omega^{(l)}$ and continuous eigenmodes $u^{(l)}(x)$, defining the relevant function space $H^1_{E_0}$. Using a finite element space $S^h_0 \subset H^1_{E_0}$, containing piecewise linear finite elements on a uniform mesh with spacing $1/N$, deduce the Rayleigh-Ritz equations giving approximate frequencies $\Omega^{(l)}$ and eigenmodes $U^{(l)}(x)$.

Define the Rayleigh Quotient, and state the Courant minimax principle characterising the eigenvalues $\lambda_l \equiv (\omega^{(l)})^2$ of the system, written in the form $a(u^{(l)}, v) = \lambda_l(u^{(l)}, v) \ \ \forall v \in H^1_{E_0}$.

Let $B_m = \text{span}\{u^{(1)}, u^{(2)}, \dots u^{(m)}\}$ where $u^{(1)}, , u^{(2)}, \dots u^{(m)}$ are the first m eigenvectors, corresponding to $\lambda_1 < \lambda_2 < \dots < \lambda_m$; and suppose that PB_m has dimension m, where $P : H^1_{E_0} \to S^h_0$ is the projection onto the finite element space defined by

$$a(v - Pv, W) = 0 \quad \forall W \in S^h_0.$$

Show that if $\Lambda_m \equiv (\Omega^{(m)})^2$ is the finite element approximation to λ_m, then

$$\lambda_m \leq \Lambda_m \leq \lambda_m \max_{v \in B_m} (\|v\|^2_{L_2}/\|Pv\|^2_{L_2}).$$

6. Irrotational, incompressible flow in a curved two dimensional channel is to be approximated on a mesh of (straight-sided) quadrilateral finite elements. If the velocity potential is given by $\nabla^2\Phi = 0$, explain the physical significance of the boundary conditions $\partial\Phi/\partial n = 0$ on the channel walls, $\partial\Phi/\partial n$ given at inlet and $\Phi = 0$ across a parallel-sided outlet, and state the variational formulation of the problem. ($\partial/\partial n$ denotes differentiation in the outward normal direction.)

Derive the mapping from local to global co-ordinates for a quadrilateral and describe briefly how this is used to calculate the stiffness matrix obtained from setting up the discrete variational problem for the finite element approximation $U(x, y)$ of Φ.

Derive the tensor product local basis functions for both a bilinear and a biquadratic approximation $U(x, y)$. Comment briefly on (i) the need for Gaussian quadrature formulae; and (ii) how the curved boundaries might be better approximated.