DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A1 Mathematical Methods I

HILARY TERM 2017

THURSDAY, 12 JANUARY 2017, 9.30am to 11.30am

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question in a new booklet.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Applied Partial Differential Equations

1. (a) [12 marks] Consider the PDE system

$$\begin{aligned} &(1+u)u_x + yu_y = u, \\ &u(x,1) = \beta x, \quad 0 \leqslant x \leqslant 1. \end{aligned} \tag{1}$$

(i) Find a sufficient and necessary condition on the **constant** β such that characteristic projections do not intersect.

(ii) Sketch the region (in the x-y plane) where the solution is uniquely defined in the case $\beta = 1$.

(b) [13 marks] Suppose that $u = u(x, y, z)$ satisfies

$$u_x + u_y + u_z = u,$$

with $u = x$ on $x + y + 2z = 1$ for $x \geqslant 1$. Use the method of characteristics to obtain an explicit solution $u(x, y, z)$.

What is the domain of definition?

2. (a) [7 marks] Consider the following PDE for $u(x, y)$:

$$F(p, q) = 0,$$

where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$. Derive Charpit's equations for this system:

$$\dot{x} = F_p, \quad \dot{y} = F_q, \quad \dot{u} = pF_p + qF_q, \quad \dot{p} = 0, \quad \dot{q} = 0, \tag{2}$$

where an overdot represents differentiation with respect to τ.
Show that all characteristics (rays) are straight lines in the x-y plane.

(b) [8 marks] Let $F = p^3 + q^4 - 2$. Suppose that $(x, y) = (1, 0)$ is on the boundary curve Γ, and that, at this point, $p_0 = q_0 = 1$ and $u_0 = 0$. Assuming that rays do not intersect, find the value of u at the point $(x, y) = (4, 4)$.

(c) [10 marks] A plane wave of light incident from $x = +\infty$ reflects off the parabola $y^2 = 4x$. The phase $u(x, y)$ of the reflected wave

$$\phi_R = Ae^{iu(x,y)}$$

satisfies the Eikonal equation $|\nabla u|^2 = 1$ and boundary condition $u = -x$ on the parabola. Find the path of the reflected rays and hence show that all rays intersect at the focus $(1, 0)$.

3. (a) [8 marks] By transforming to a first order system, obtain a quadratic equation for the characteristics $\lambda = \frac{dy}{dx}$ of the PDE

$$a(x,y)\phi_{xx} + b(x,y)\phi_{xy} + c(x,y)\phi_{yy} = f(x,y).$$

What does it mean for the system to be hyperbolic?

[*You may use without proof that characteristic curves* $(x(\tau), y(\tau))$ *of the system* $\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y = \mathbf{c}$ *satisfy*

$$\det\left(\frac{dx}{d\tau}\mathbf{B} - \frac{dy}{d\tau}\mathbf{A}\right) = 0.]$$

(b) [17 marks] Consider the PDE system, defined for $u > 0$:

$$\begin{aligned} \frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u^2\frac{\partial v}{\partial y} &= 0, \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} &= 0, \end{aligned} \tag{3}$$

(i) Show that (3) is a hyperbolic system, and that it has Riemann invariants $\log u \pm v$ along appropriate characteristics whose direction you should find.

(ii) Given the boundary data

$$u = u_0, v = v_0 \quad \text{on} \quad y = 0, x > 0$$

where $0 < v_0 < u_0$ are **constants**, give the domain of definition of the solution.

4. (a) [12 marks] Consider the heat equation on a half-line:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad t > 0, x > 0 \\ &u(x,0) = g(x), \\ &u(0,t) = h(t), \\ &u(x,t) \to 0 \quad \text{as } x \to \infty \end{aligned} \tag{4}$$

(i) State the problem satisfied by the Green's function $G(x,t)$ for this system and obtain an expression for the solution to (4) in terms of G.

(ii) Give an explicit form for $G(x,t)$.
[*You may use without proof that*

$$y(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}$$

satisfies

$$\begin{aligned} &\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \\ &y(x,0) = \delta(x), \\ &y(x,t) \to 0 \quad \textit{as } x \to \pm\infty.] \end{aligned}$$

(b) [13 marks] Consider the parabolic equation

$$x^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } x > 0, t > 0$$

subject to $u(x,0) = 0$, $u(0,t) = t^m$, $u \to 0$ as $x \to \infty$, where m is a constant.
Show that the system admits a similarity solution

$$u(x,t) = t^\alpha f(\eta) \quad \text{with } \eta = \frac{x}{t^\beta},$$

where α and β are constants that you should determine.
Hence determine the ODE and boundary conditions satisfied by $f(\eta)$.

Section B: Supplementary Mathematical Methods

5. The differential operator L is defined by

$$Ly \equiv y''(x) - 2y'(x) + y(x) \tag{5}$$

on $0 < x < \pi$.

(a) [10 marks] Find the eigenvalues λ and corresponding eigenfunctions y for

$$Ly = \lambda y$$

with boundary conditions

$$y(0) - y'(0) = 0, \quad y(\pi) = 0. \tag{6}$$

[You may use, without proof, that all eigenvalues are strictly negative.]

(b) (i) [6 marks] Find an appropriate function $r(x)$ so that $\hat{L} \equiv rL$ is a Sturm-Liouville operator. Give the eigenvalues $\hat{\lambda}$ and eigenfunctions $\hat{y}$ for $\hat{L}\hat{y} = \hat{\lambda} r \hat{y}$ with boundary conditions

$$\hat{y}(0) - \hat{y}'(0) = 0, \quad \hat{y}(\pi) = 0;$$

(ii) [9 marks] Use this to obtain a formula for the coefficients in an eigenfunction expansion

$$y = \sum_{n=0}^{\infty} c_k \hat{y}_k(x)$$

for the solution of the problem

$$y''(x) - 2y'(x) + y(x) = x,$$

with boundary conditions

$$y(0) - y'(0) = 0, \quad y(\pi) = b,$$

where b is a real constant.

[You do not need to evaluate the integrals in the formula for c_k.]

6. (a) (i) [5 marks] Find the general solution of the linear differential equation:

$$Ly \equiv 2x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \tag{7}$$

for $1 < x < \mathrm{e}$.

(ii) [10 marks] Consider the boundary value problem

$$Ly(x) = f(x), \quad 1 < x < e, \qquad y(1) = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(\mathrm{e}) = 0, \tag{8}$$

with Ly as in (7). Write down two equivalent problems for the Green's function $g(x,\xi)$:
(I) using the delta function $\delta(x)$;
(II) using only classical functions and with appropriate conditions at $x = \xi$.
Determine $g(x,\xi)$ explicitly.

(b) [10 marks] Let

$$f(x) = \sum_{j=-\infty}^{\infty} g(x - 2j),$$

where

$$g(x) = \begin{cases} x & \text{if } -1 \leqslant x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

State the definition of the derivative of a distribution, and use it to show that the derivative of f *in the distributional sense* is

$$Df = \alpha + \sum_{j=-\infty}^{\infty} \beta\, T(x - 2j - 1),$$

with a distribution T and constants α and β, all of which you are to determine.