JMAT 7301

M.Sc. in Mathematical Modelling and Scientific Computing
MSc in Applied and Computational Mathematics

Paper A (Mathematical Methods)

Thursday 19 April, 2001, 9.30 a.m. - 12.30 p.m.

Candidates may attempt as many questions as they wish. All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Mathematical Methods I

1. Show that

$$L[u] \equiv -u'' = 0, \qquad 0 \leq x \leq 1,$$

with the boundary conditions $u(0) - u'(0) = 0$, $u(1) - u'(1) = 0$, forms a self-adjoint system. Calculate the eigenvalues and eigenfunctions of $L[u] = \lambda u$.

State sufficient conditions on f for there to be a uniformly convergent expansion

$$f(x) = \sum_{n=1}^{\infty} a_n(\sin n\pi x + n\pi \cos n\pi x) \qquad 0 \leq x \leq 1,$$

and express a_n as an integral in terms of f.

2. Consider the non-linear oscillator equation

$$\frac{d^2x}{dt^2} + k(x^2 - x - 2)\frac{dx}{dt} + x = 1.$$

Show that in the limit $k \ll 1$ the closed trajectory has period $2\pi + O(k^2)$, and that its leading-order approximation is $x_0 = 1 + 2\sqrt{2}\cos t$. [*You may use the fact that* $\cos^2 y \sin y = \frac{1}{4}(\sin y + \sin 3y)$.]

Find $F(x)$ such that the system may be written in Liénard form

$$\frac{dx}{dt} = k(y - F(x)), \tag{1}$$

$$k\frac{dy}{dt} = 1 - x. \tag{2}$$

In the limit $k \gg 1$ sketch the trajectories in the xy-plane, and show that the period of the closed trajectory is approximately

$$k\left(\frac{27}{4} - 2\log(35/8)\right).$$

3. A process obeys the first-order differential equation

$$\dot{x} = -x^3 + u^3,$$

where u is an unrestricted control. It is desired to minimise, for a given $x(0) > 0$ and $T > 0$, the integral

$$I = \int_0^T (x^4 + u^4)\,dt.$$

Find the Hamiltonian, explaining the procedure you use, show that the optimal value of u is $-3p/4$, where p is the adjoint variable, and write down the differential equation for p. Show that the differential equations are satisfied by an optimal feedback-control law in the form

$$u(t) = -K(x(t))x(t),$$

where the gain (which is a function of x only) satisfies the differential equation

$$x\frac{dK}{dx} = \frac{3 - 4K - K^4}{1 + K^3}$$

Specify a boundary value for K and hence shown that K is given implicitly by

$$x = 3^{1/4}x(T)(3 - 4K - K^4)^{-1/4}.$$

Using the differential equation for x show that K is given implicitly as a function of t by

$$T - t = \frac{1}{3^{1/2}x(T)^2}\int_0^K \frac{dk}{(3 - 4k - k^4)^{1/2}}.$$

4. Define a *weak solution* of the partial differential equation

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$$

where P and Q are differentiable functions of x, y and u. Show that if C is a curve in the xy-plane and u is a weak solution which takes different values on either side of C, then along C

$$\frac{dy}{dx} = \frac{[Q]}{[P]}. \qquad (*)$$

Show, by giving an example, that two related partial differential equations may have the same continuous solutions and yet have different shock relations $(*)$.

An initial value problem for $u(x,\, y)$ is defined by

$$(1+u)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

with $u(x,\, 0) = f(x)$. Show that an implicit form of the solution is given by

$$u = f(x - (1+u)y)$$

and explain why this form of the solution may not lead to a unique answer. Illustrate your answer by considering

$$f(x) = \begin{cases} 1 \text{ if } x < 0 \\ -1 \text{ if } x > 0 \end{cases}$$

and finding two different weak solutions which are both valid for $y > 0$. If y is time, explain how you can decide which, if any, of your solutions is physically viable?

5. The Riemann Function, G, for the partial differential equation

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = f(x, y)$$

is defined as follows:

$$\mathcal{L}^* G = \frac{\partial^2 G}{\partial x \partial y} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + u = 0 \text{ if } x \neq \xi,\, y \neq \eta,$$

$$\frac{\partial G}{\partial x} = G \text{ on } y = \eta$$

$$\frac{\partial G}{\partial y} = G \text{ on } x = \xi$$

$$\text{and } G = 1 \text{ at } x = \xi,\, y = \eta.$$

Show that

$$G\mathcal{L}u - u\mathcal{L}^* G = \frac{\partial}{\partial x}(G\frac{\partial u}{\partial y} + Gu) - \frac{\partial}{\partial y}(u\frac{\partial G}{\partial x} - Gu),$$

and deduce that, if Cauchy data is given on the line $x + y = 0$, then

$$u(\xi, \eta) = \int_{-\xi}^{\eta} \int_{-y}^{\xi} G f dx dy + \int_{-\xi}^{\eta} \left[G \left(u + \frac{\partial u}{\partial y} \right) \right]_{x=-y} dy$$

$$- \int_{-\eta}^{\xi} \left[u \left(\frac{\partial G}{\partial x} - G \right) \right]_{y=-x} dx + [uG]_{x=\xi,\, y=-\xi}.$$

Show that $G = e^{x+y-\xi-\eta}$. Hence show that if $f = 0$ with Cauchy data $u = 0$, $\frac{\partial u}{\partial x} = \beta(x)$ on $x + y = 0$, the solution is given by $u(\xi, \eta) = e^{-\eta-\xi} \int_{-\eta}^{\xi} \beta(\tau) d\tau$.

Suppose that $\beta(x)$ is continuous and has continuous derivatives everywhere except at $x = 0$ where there is a jump in $\beta'(x)$. Describe the type and location of any resulting discontinuities in the solution for u.

6. (i) Show that the system of equations

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + \mu \frac{\partial u}{\partial x} = 0$$

can be written as a scalar second order equation for u and that the problem is hyperbolic if $\mu > 0$ and elliptic if $\mu < 0$.

Suppose that u, p are given on $t = 0$. State the Cauchy-Kowalesky Theorem and explain how it applies to this problem. Does the sign of μ make any difference to what the theorem says about solutions of this problem?

When $\mu = -1$, solve for u, p given that

$$u = \cos x \text{ and } p = 0 \text{ on } t = 0.$$

Is it possible to find a solution that is bounded everywhere? Would the answer to this question be the same if $\mu = 1$? Justify your assertions.

(ii) The function $Z(x, t)$ satisfies the equation

$$Z_t = Z_{xx}$$

in the region D given by $-1 < x < 1$, $0 < t < T$. If Z is given on $x = \pm 1$ for $0 < t < T$ and on $t = 0$ for $-1 < x < 1$, prove that Z will take its maximum value in D either on $x = \pm 1$ or on $t = 0$.

Hence show that there is at most one solution of the equation

$$Z_t = Z_{xx} + f(x, t)$$

when Z is subject to the same boundary conditions and f is a continuous function of x, t.