JMAT 7301
JACM 7301
JACM 7C61
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JMAT 7301

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods

Thursday, 22nd April 2004, 9:30 a.m. – 12:30 p.m.

Candidates may attempt as many questions as they wish.

JACM 7301

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods I & II

Thursday, 22nd April 2004, 9:30 a.m. – 12:30 p.m.

Candidates may attempt as many questions as they wish.

JACM 7C61

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods I

Thursday, 22nd April 2004, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 1,2,3 only.

JACM 7C62

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods II

Thursday, 22nd April 2004, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 4,5,6 only.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Mathematical Methods I

Question 1

Consider the differential equation for u,

$$x\, u_{xx} + u_x = f \quad \text{for} \quad 0 < x < 1, \tag{1}$$

where $u_x := \mathrm{d}u/\mathrm{d}x$ and f is a real continuous function defined on $[0,1]$, together with the boundary conditions

$$u_x(0) \text{ is bounded}, \qquad u_x(1) + a\, u(1) = 0, \tag{2}$$

where a is a real constant.

(i) In the particular case $a = 0$ and $f(x) = 4x^3 - 2x$, find the general solution of equations (1) and (2).

[4 marks]

(ii) Let $\delta(x - s)$ be the Dirac delta function. For the case $a \neq 0$, find the Green's function $G(x,s)$ for $x, s \in [0,1]$, satisfying

$$(xG_x)_x = \delta(x - s) \quad \text{for} \quad 0 < x < 1, \tag{3}$$
$$G_x(0,s) \text{ is bounded}, \qquad G_x(1,s) + a\, G(1,s) = 0,$$

for all $s \in [0,1]$. [6 marks]

Show that your solution $G(x,s)$ satisfies (3) in the sense of distributions. [5 marks]

Hence show that, for $a \neq 0$, the general solution of equations (1) and (2) can be written in terms of an integral involving $G(x,s)$. Why is your solution unique? [4 marks]

(iii) For $a = 0$, use the homogeneous adjoint problem to find a condition on the function f that guarantees equations (1) and (2) have a solution. Verify that $4x^3 - 2x$ satisfies your condition. [6 marks]

Question 2

(i) Show that the partial differential equation

$$v_t - v^2 v_x = k v_{xx} + v + \lambda$$

for real constants $\lambda \neq 0$ and $k > 0$ has travelling wave solutions of the form $v(x,t) = -\lambda\, u(z)$, $z = (x+ct)/\sqrt{k}$ if $u(z)$ satisfies the ordinary differential equation

$$u'' + u - 1 = \epsilon\,(\alpha - u^2)u' \qquad (1)$$

where $u' := \mathrm{d}u/\mathrm{d}z$, and α and ϵ are constants which you should find in terms of k, λ and c. [6 marks]

(ii) Show that when $\epsilon \ll 1$ and α lies in an interval which you should determine, there is a periodic solution of equation (1) of the form

$$u(z) = 1 + A\sin\omega z + \epsilon\, u_1(z) + O(\epsilon^2),$$

where the period of $u(z)$ is $2\pi + O(\epsilon^2)$, and $u(0) = 1$. Find A in terms of α. [5+4 marks]

[You may use the identity $\sin^2 z\ \cos z = \frac{1}{4}(\cos z - \cos 3z)$.]

Find the form of $u_1(z)$ up to undetermined constants, and give a brief explanation of how you could find these constants and the $O(\epsilon^2)$ correction to the period. [5 marks]

(iii) Consider the (u, u') phase plane for equation (1), find any equilibrium points and classify their stability. Hence, suggest why such a limit cycle (or periodic orbit) might exist when $\alpha > 1$ and $\epsilon > 0$ is sufficiently small. [5 marks]

Question 3

Consider the differential equation, for small $\epsilon > 0$,

$$\epsilon^2 y'' - (1-x)y' + \frac{1}{2}(1-x)y^2 = 0, \qquad \text{for} \quad 0 < x < 1,$$

where $y' := \mathrm{d}y/\mathrm{d}x$, with boundary conditions

$$y(0) = 1 \quad \text{and} \quad y(1) = 0.$$

(i) Show that there cannot be a boundary layer at $x = 0$. [7 marks]

(ii) Find the leading order behaviour of $y(x)$ at fixed $x \in (0,1)$ as $\epsilon \to 0$. [5 marks]

(iii) Find the leading order boundary layer behaviour of $y(x)$ near $x = 1$, and hence find a composite solution that yields a uniformly valid leading order approximation to $y(x)$ for $0 \le x \le 1$. [6+5 marks]

(iv) Hence, give a rough sketch of the solution for $\epsilon = 0.1$. [2 marks]

Mathematical Methods II

Question 4

Find the explicit solutions $u(x,y)$ of the following PDE problems and, in each case, determine the region of the (x,y) plane where the solution is well defined.

(a) $(3x+4y)\dfrac{\partial u}{\partial x}+(4x-3y)\dfrac{\partial u}{\partial y}=5u, \qquad u(x,0)=1, \ -\infty<x<\infty;$ [8 marks]

(b) $u^2\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=0, \qquad u(x,0)=-x, \ -\infty<x<\infty;$ [8 marks]

(c) $\left(\dfrac{\partial u}{\partial x}\right)^2\dfrac{\partial u}{\partial y}=1, \qquad u(x,0)=x^{2/3}, \ 0<x<1.$ [9 marks]

Question 5

Consider the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} &= 0, \quad (x,y)\in D, \\ \alpha(x,y)u+\beta(x,y)\frac{\partial u}{\partial n} &= g(x,y), \quad (x,y)\in\partial D, \end{aligned} \qquad (\star)$$

where D is a closed bounded region of the (x,y)-plane with smooth boundary ∂D and outward-pointing unit normal $\boldsymbol{n}$.

(i) Show that, if a solution of ($\star$) exists, then it is unique provided $\alpha>0$ and $\beta\geq 0$. [5 marks]

(ii) For the case $\alpha\equiv 0$, $\beta\equiv 1$, show that ($\star$) has no solution unless a *solvability condition* (which you should find) is satisfied, in which case the solution is non-unique. [5 marks]

(iii) Now suppose that $\beta=1$ and D is the unit circle $x^2+y^2\leq 1$. Show that nonzero solutions to the homogeneous problem (with $g=0$) exist if $\alpha=-n$, where n is a positive integer. [*Hint: look for a separable solution in polar coordinates.*] [5 marks]

(iv) For the case $\alpha>0, \beta\geq 0$, show that, if the function $G(x,y;\xi,\eta)$ is defined to satisfy

$$\begin{aligned} \frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2} &= 0, \quad (x,y)\in D\setminus(\xi,\eta), \\ \alpha(x,y)G+\beta(x,y)\frac{\partial G}{\partial n} &= 0, \quad (x,y)\in\partial D, \\ G\sim\frac{1}{2\pi}\log|(x,y)-(\xi,\eta)| & \quad \text{as} \quad (x,y)\to(\xi,\eta), \end{aligned}$$

then the solution of ($\star$) is

$$u(\xi,\eta)=\oint_{\partial D}\left(\frac{g}{\alpha}\right)\frac{\partial G}{\partial n}\,\mathrm{d}s.$$

[10 marks]

Question 6

(a) A function $u(x, y)$ satisfies

$$\iint_D \left(P(x, y, u)\frac{\partial \psi}{\partial x} + Q(x, y, u)\frac{\partial \psi}{\partial y} + R(x, y, u)\psi \right) \mathrm{d}x \, \mathrm{d}y = 0$$

for all continuously differentiable test functions ψ that vanish on the boundary of the region D. Show that, if P and Q are continuously differentiable, and u is continuously differentiable in D, then u must satisfy the PDE

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = R.$$

Show also that, if u is continuously differentiable everywhere except on a curve C that crosses D, then the slope of C must be given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{[Q]_-^+}{[P]_-^+},$$

where $[\,\cdot\,]_-^+$ denotes the jump in the value of $\,\cdot\,$ across C. [10 marks]

(b) Consider the PDE system

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uw) &= 0, \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(vw) &= 0, \\ \frac{\partial}{\partial t}(uw) + \frac{\partial}{\partial x}\left(uw^2 + v\right) &= 0. \end{aligned}$$

(i) Show that the system is hyperbolic provided $v/u > 0$. [4 marks]

(ii) Show that the Riemann invariant $R = v/u$ is conserved along the characteristics $\mathrm{d}x/\mathrm{d}t = w$. [5 marks]

(iii) Now assuming that $R \equiv c^2$, where c is a constant, find the other two Riemann invariants. [6 marks]