JMAT 7301
JACM 7301
JACM 7C61
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JMAT 7301

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods

Thursday, 21st April 2005, 9:30 a.m. – 12:30 p.m.

Candidates may attempt as many questions as they wish.

JACM 7301

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods I & II

Thursday, 21st April 2005, 9:30 a.m. – 12:30 p.m.

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JACM 7C61

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods I

Thursday, 21st April 2005, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 1,2,3 only.

JACM 7C62

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods II

Thursday, 21st April 2005, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 4,5,6 only.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Techniques of Mathematical Modelling I

Question 1

(i) Explain what is meant by a *regular perturbation* and by a *singular perturbation* for an algebraic equation and for an ordinary differential equation. [*It may be useful to give illustrative examples*.]
[3 marks]

(ii) The quantity x satisfies the algebraic equation

$$x^5 - \varepsilon x - 1 = 0. \qquad (\star)$$

Suppose that $\varepsilon \ll 1$. Find approximate expressions, correct to terms of $O(\varepsilon)$, for each of the five solutions of the equation. [7 marks]

(iii) Now suppose $\varepsilon \gg 1$. Show that by a suitable rescaling of the solution, which you should find, ($\star$) can be written in the form

$$X^5 - X - \delta = 0,$$

where $\delta = \varepsilon^{-5/4} \ll 1$.

Hence find leading order (non-zero) approximations for all five of the solutions to ($\star$). [11 marks]

(iv) Find the next order approximation to the smallest root in this case, $\varepsilon \gg 1$. [4 marks]

Question 2

A population $u(x,t)$ satisfies

$$u_t = ru\left(1 - \frac{u}{K}\right) + \{Du_x\}_x \,.$$

(i) Assuming that r, K and D are constant, show how to non-dimensionalise the equation to the form

$$u_t = u\,(1-u) + u_{xx}.$$

[2 marks]

(ii) If, initially, $u=0$ except on a finite interval where it is small and positive, explain using diagrams how you would expect the solution to evolve.

Verify your description by seeking travelling wave solutions of the form $u=f(\xi)$, $\xi=x-ct$, and write down the resulting equation and boundary conditions which f must satisfy, assuming that $c > 0$. How should the boundary conditions be modified if $c < 0$?

[8 marks]

(iii) By examining the solutions in a suitable phase plane, show that a travelling wave in which $u > 0$ is possible if $c \geq 2$. What happens if $c < 2$? [12 marks]

(iv) If, instead, $D(u) = D_0 u$, what form would you expect the evolution to take? [3 marks]

Question 3

(a) The C^2 function $y(x)$, satisfying $y(0) = y(1) = 0$, is an extremal for

$$\int_0^1 F(x, y, y')\,\mathrm{d}x.$$

Show that y satisfies the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0.$$

What extra condition is required if $y(1)$ is not specified? [8 marks]

(b) Define the delta distribution $\delta(x)$ by its action $\langle \delta, \phi \rangle$ on a test function $\phi(x)$ and prove that it is the derivative of the Heaviside function (which is equal to 1 if $x > 0$, and to 0 if $x < 0$).

Suppose that

$$f_\epsilon(x) = \frac{1}{\epsilon\sqrt{2\pi}}\,\mathrm{e}^{-x^2/(2\epsilon^2)}.$$

Explain why, as $\epsilon \to 0$, the action of f_ϵ on a test function $\phi(x)$ tends to that of $\delta(x)$ on $\phi(x)$.

[8 marks]

(c) The function $u(x, t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}, \qquad -\infty < x < \infty, \quad t > 0,$$

with the initial condition $u(x, 0) = \delta(x)$. By taking a Fourier Transform in x, or otherwise, show that

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(x-t)^2}{4t}\right).$$

[9 marks]

Techniques of Mathematical Modelling II

Question 4

(i) Define a weak solution of the system of partial differential equations

$$\frac{\partial \boldsymbol{P}}{\partial t} + \frac{\partial \boldsymbol{Q}}{\partial x} = \boldsymbol{R},$$

where $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are vector-valued differentiable functions of x, t and $\boldsymbol{u}$.

Show that if C is a curve in the xt-plane and $\boldsymbol{u}$ is a weak solution which takes different values on either side of C then along C

$$[\boldsymbol{P}]\,\frac{\mathrm{d}x}{\mathrm{d}t} = [\boldsymbol{Q}]\,,$$

where $[\cdot]$ denotes the jump across C. [7 marks]

(ii) Show that the system

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uv) &= 0, \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} + \frac{v^2}{2}\right) &= 0, \end{aligned}$$

is hyperbolic, and find its characteristics and Riemann invariants.

Deduce that the characteristics are straight lines. [6 marks]

(iii) Consider the solution of the partial differential equations in (ii), subject to the initial data

$$u(x,0) = \begin{cases} 2, & x > 0, \\ 1, & x < 0 \end{cases} \qquad v(x,0) = \begin{cases} 0, & x > 0, \\ 3, & x < 0. \end{cases}$$

Show that there is no single shock solution in which

$$u(x,t) = \begin{cases} 2, & x > c(t), \\ 1, & x < c(t) \end{cases} \qquad v(x,t) = \begin{cases} 0, & x > c(t), \\ 3, & x < c(t). \end{cases}$$

Use the Rankine-Hugoniot conditions to show that a double-shock solution in the form

$$u(x,t) = \begin{cases} 2, & x > 3t, \\ u_0, & 0 < x < 3t, \\ 1, & x < 0 \end{cases} \qquad v(x,t) = \begin{cases} 0, & x > 3t, \\ v_0, & 0 < x < 3t, \\ 3, & x < 0, \end{cases}$$

is possible, and determine the values of u_0 and v_0.

Sketch the characteristics in each region. Does this solution obey causality? [12 marks]

Question 5

Suppose that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u,$$

and that $u(x,y)$ satisfies the boundary data

$$u(1,y) = 1 \text{ for } y \leq 0, \quad u(x,y) = 1 \text{ for } x^2 + y^2 = 1, y > 0.$$

Find all the solutions that are differentiable away from the boundary and the negative x-axis (including the origin). [25 marks]

Question 6

Let the twice continuously differentiable function $u(x,t)$ satisfy the partial differential equation

$$u_t = x\, u_{xx} - f(x,t,u), \text{ in } x,t > 0,$$

where $u(0,t) = 0$ for $t > 0$, and $u(x,t) \to 0$ for $x \to \infty, t > 0$.

(i) If $f(x,t,u) > 0$, show that $u(x,t)$ cannot have a maximum at any point (x_0, t_0) with $x_0, t_0 > 0$.

If, in addition, $u(x,0) \leq 0$, show that $u(x,t) < 0$ for all $x,t > 0$. [13 marks]

(ii) Suppose now that $f(x,t,u) = x\,\mathrm{e}^{-x}$. Find a solution of the above problem such that $u(x,0) = \mathrm{e}^{-x}$, and deduce that, as $t \to \infty$, $u \to \mathrm{e}^{-x} - 1$.

[Hint: write $u = \mathrm{e}^{-x} + v$ and find a similarity solution for v.] [12 marks]