JMAT 7301
JACM 7301
JACM 7C61
JACM 7C62

TRINITY TERM 2006

JMAT 7301

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods

Thursday, 20th April 2006, 9:30 a.m. – 12:30 p.m.

Candidates may attempt as many questions as they wish.

JACM 7301

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods I & II

Thursday, 20th April 2006, 9:30 a.m. – 12:30 p.m.

Candidates may attempt as many questions as they wish.

JACM 7C61

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods I

Thursday, 20th April 2006, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 1,2,3 only.

JACM 7C62

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods II

Thursday, 20th April 2006, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 4,5,6 only.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Techniques of Mathematical Modelling I

Question 1

(a) Find leading order approximations to the solution of the boundary value problem

$$\epsilon y'' + (x^2+1)y' - x^3 y = 0,$$

$$y(0) = y(1) = 1,$$

assuming that $\epsilon \ll 1$, describing where your approximations are valid.

(8 marks)

(b) The function $W(\xi)$ is defined by

$$We^W = \xi.$$

Show, graphically or otherwise, that W is a monotonically increasing positive function of ξ when ξ is positive.

Show that $W \sim \xi$ as $\xi \to 0$, and explain why $W \sim \ln\xi - \ln\ln\xi$ as $\xi \to \infty$. (7 marks)

(c) The function $y(x)$ is determined by the solution of the boundary value problem

$$\epsilon y'' + (y^2+1)y' - x^3 y = 0,$$

$$y(0) = y(1) = 1,$$

where $\epsilon \ll 1$.

Find an outer approximation y_{out}, valid away from $x = 0$, to the solution, and show that as $x \to 0$,

$$y_{\text{out}} \to \sqrt{W(e^{1/2})} \equiv y_\infty.$$

Find a suitable approximating equation for the solution in the boundary layer near $x = 0$, and write down the boundary conditions which its solution must satisfy. Hence show that the equation can be integrated to the form

$$\frac{dy}{dX} = p(y_\infty) - p(y)$$

for a suitable function $p(y)$ and boundary layer coordinate X, and deduce (by considering the graph of $p(y)$) that a solution to the boundary layer equation exists. (It is not necessary to find this solution.)

(10 marks)

Question 2

The subsidence rate $w(X,t)$ of an ice cauldron on an ice sheet is governed by the beam equation

$$w_{XXXX} = -N,$$

where the effective load $N(X,t)$ satisfies

$$N_t = \dot{N}_+ + w,$$

and $N_+(t)$ is a function of t ($\dot{N}_+$ being its derivative). The model is to be solved on $0 < X < \infty$, and we prescribe

$$w = w_{XX} = 0, \quad N = N_+, \quad \text{on} \quad X = 0,$$

$$w \to -\dot{N}_+ \quad \text{as} \quad X \to \infty.$$

(a) By writing $w = -\dot{N}_+ + W$ and integrating repeatedly, show that

$$W = \int_X^\infty \tfrac{1}{6}(\xi - X)^3 N(\xi,t)\,d\xi,$$

and deduce that

$$\dot{N}_+ = \int_0^\infty \tfrac{1}{6}\xi^3 N(\xi,t)\,d\xi.$$

Hence show that N satisfies the integro-differential equation

$$N_t = \dot{N}_+ + \int_0^\infty G(X,\xi)N(\xi,t)\,d\xi,$$

and give the definition of G.

(11 marks)

(b) Show directly from the governing equations that if

$$N_+ = \frac{c}{(t_0 - t)^\alpha},$$

there is a similarity solution of the form

$$N = N_+\psi(\eta), \quad W = \dot{N}_+\phi(\eta),$$

where

$$\eta = mX(t_0 - t)^\beta,$$

and β should be determined. Show that, by choosing the value of m suitably, the equation for ϕ can be written in the form (after eliminating ψ)

$$\epsilon\eta\phi^{\text{v}} - \phi^{\text{iv}} - \phi = 0,$$

where the Roman numeral superscripts indicate the number of derivatives. Give the value of ϵ, and write down suitable boundary conditions for ϕ. **(14 marks)**

Question 3

(a) State the definitions of a compact-support test function of one variable, and of a distribution $\mathcal{D}$. Define the Heaviside function $\mathcal{H}(x)$ and the delta function $\delta(x)$ in terms of their actions on test functions. Define the derivative $\mathcal{D}'$ of $\mathcal{D}$ in terms of the action of $\mathcal{D}$ on a test function $\phi(x)$. Show that

$$\mathcal{D}'(x) = \lim_{h\to 0} \frac{\mathcal{D}(x+h) - \mathcal{D}(x)}{h}.$$

Use the right-hand side of this equation to show directly that the derivative of the Heaviside function is the delta function. **(9 marks)**

(b) Calculate the Fourier transform of $f(x) = \mathrm{e}^{-\epsilon|x|}$, defined by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-\epsilon|x|}\mathrm{e}^{ikx}\,\mathrm{d}x.$$

By considering $\lim_{\epsilon\to 0} \widehat{f}(k)$, calculate the Fourier transform of 1. **(6 marks)**

(c) Derive the Euler–Lagrange equation for minimisers of the functional

$$I[u] = \iint_D \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \mathrm{d}x\mathrm{d}y$$

over all twice continuously differentiable functions $u(x, y)$ whose values are equal to a given function g on the boundary of the finite simply connected region D in $\mathbb{R}^2$.

Find $u(x, y)$ when D is the unit disc $x^2 + y^2 < 1$, and $u(x, y) = 2x^2 - 1$ on ∂D. **(10 marks)**

Techniques of Mathematical Modelling II

Question 4

(a) Use Charpit's method to find, in parametric form, the two solutions $u(x, y)$ of the equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u^2$$

with the boundary data $u = 1$ on $x^2 + y^2 = 1$. State clearly where each solution is defined uniquely by the data, and sketch the characteristic projections in the (x, y) plane. **(15 marks)**

(b) If the part of the data in part (a) where $y < 0$ is altered so that $u(1, y) = 1$ for $y < 0$, find the new solutions for $y < 0$, state where they are defined uniquely by the new data, and state where on the x axis these new solutions are continuous with the previous solutions. **(10 marks)**

[*You may assume that Charpit's equations for the equation* $F(x, y, u, p, q) = 0$ *are*

$$\dot{x} = F_p, \quad \dot{y} = F_q, \quad \dot{u} = pF_p + qF_q,$$
$$\dot{p} = -F_x - pF_u, \quad \dot{q} = -F_y - qF_u,$$

where $\dot{} = \mathrm{d}\,/\mathrm{d}t$ *and subscripts indicate partial derivatives.*]

Question 5

Consider the partial differential equation

$$u_t = (uu_x)_x.$$

(a) Write this equation in canonical form, state why it is parabolic, and give the characteristics.

(6 marks)

(b) Now suppose that $u(x, t)$ represents the density of a substance, and consider smooth positive solutions of the partial differential equation above for $-x_f(t) < x < x_f(t), \quad t > 1$, with the boundary conditions

$$u(\pm x_f(t), t) = 0, \qquad t \geq 1,$$

and suppose that $u(x, 1)$ is a given positive even function of x. Show by considering

$$\int_{-x_f(t)}^{x_f(t)} u(x, t)\, \mathrm{d}x$$

that the total amount of the substance between $x = -x_f(t)$ and $x = x_f(t)$ is conserved, stating any properties of the solution that you use.

(5 marks)

(c) Suppose that $x_f(t) = t^\gamma$, where γ is a positive constant. Show that there are similarity solutions of the form

$$u(x, t) = t^\alpha f(x/t^\beta)$$

provided that $\gamma = \frac{1}{3}$ and the constants α, β take suitable values which you should find. Deduce that $f(\eta)$, where $\eta = x/t^\beta$, satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f\frac{\mathrm{d}f}{\mathrm{d}\eta}\right) = -\frac{1}{3}f - \frac{1}{3}\eta\frac{\mathrm{d}f}{\mathrm{d}\eta}, \qquad -1 < \eta < 1,$$

and find $f(\eta)$. Deduce that $u(x, 1) = \frac{1}{6}(1 - x^2)$ for $-1 < x < 1$.

(10 marks)

(d) If $u(x, t) \equiv 0$ for $|x| > x_f(t)$, how can

$$\lim_{t \downarrow 0} u(x, t)$$

be interpreted?

(4 marks)

Question 6

The vector function $\mathbf{u}(x,t)$ satisfies the equation

$$\iint_R \mathbf{u}\frac{\partial\psi}{\partial t} + \mathbf{w}\frac{\partial\psi}{\partial x} + \mathbf{c}\psi \,\mathrm{d}x\mathrm{d}t = -\int_{-\infty}^{\infty} \psi(x,0)\mathbf{u}_0(x)\,\mathrm{d}x, \tag{1}$$

where R is the half-plane $t > 0$, $\psi(x,t)$ is any test function that is continuously differentiable and vanishes sufficiently rapidly as $|x| \to \infty$ and $t \to \infty$, and $\mathbf{w}$ and $\mathbf{c}$ are continuous functions of $\mathbf{u}$, x and t.

(a) Show that, if $\mathbf{u}$ is continuously differentiable in R, then $\mathbf{u}$ satisfies the equation

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{w}}{\partial x} = \mathbf{c}$$

in R, with $\mathbf{u}(x,0) = \mathbf{u}_0(x)$.

If $\mathbf{u}(x,t)$ is discontinuous across a curve Γ in R given by $x = x(t)$, show that

$$[\mathbf{u}]_-^+ \frac{\mathrm{d}x}{\mathrm{d}t} = [\mathbf{w}]_-^+,$$

where $[\alpha]_-^+$ denotes the jump in α from one side of Γ to the other.

(8 marks)

(b) A two-dimensional system

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} v + \frac{1}{2}\epsilon(u+v)^2 \\ u + \frac{1}{2}\epsilon(u+v)^2 \end{pmatrix}, \qquad \mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } x < 0, \quad \mathbf{u}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for } x > 0,$$

has weak solutions which satisfy (1). Show that, if u, v are continuously differentiable, the equations for u, v can be written in the form

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)(u - v) = 0, \qquad \left(\frac{\partial}{\partial t} + (1 + 2\epsilon(u+v))\frac{\partial}{\partial x}\right)(u + v) = 0.$$

Deduce that the characteristics of the system are straight lines on which

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -1 \qquad \text{or} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 1 + 2\epsilon(u+v).$$

(8 marks)

(c) With the aid of a clear diagram and a causality argument, show that there must be a discontinuity in u, v across a curve lying between $x = t$ and $x = (1+2\epsilon)t$. Write down the shock conditions for this system and show that across a discontinuity either

$$[u]_-^+ = [v]_-^+$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -1.$$

Show that there are three regions in each of which u, v are constant, and that for $t > 0$

$$\begin{array}{lll} u = 1, \quad v = 0 & \text{in} & x < -t, \\ u = \frac{1}{2}, \quad v = \frac{1}{2} & \text{in} & -t < x < (1+\epsilon)t, \\ u = 0, \quad v = 0 & \text{in} & (1+\epsilon)t < x. \end{array}$$

(9 marks)