DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A2 Mathematical Methods II

HILARY TERM 2017

THURSDAY, 20 APRIL 2017, 9.30am to 11.30am

This exam paper contains three sections.
Candidates should submit answers to a maximum of ***four*** *questions for credit that include an answer to at least* ***one*** *question in each section.*

Please start the answer to each question in a new answer booklet.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Nonlinear Systems

1. Consider the system

$$\dot{x} = x(3 - x - 5y),$$
$$\dot{y} = y(-1 + x + y).$$

with $(x, y) \in \mathbb{R}^2$.

(a) [5 marks] Show that both coordinate axes are invariant sets and that the line $J = x + 3y - 3 = 0$ is also an invariant set. Show that any intersection of these invariant sets defines a fixed point.

(b) [5 marks] Use linear analysis to determine the stability of the origin and show that there is a non-hyperbolic fixed point at $(1/2, 1/2)$.

(c) [5 marks] Find a such that $H = xy^3 J^a$ is a first integral of the system (that is, $\dot{H} = 0$).

(d) [5 marks] Use the first integral H to determine the stability of the non-hyperbolic fixed point.

(e) [5 marks] Given that the three fixed points computed in (a) are unstable and hyperbolic, sketch the phase portrait of the system.

2. Consider the tent map, mapping the unit interval $[0, 1]$ into itself and defined by

$$x_{n+1} = \begin{cases} 2x_n & 0 \leqslant x_n \leqslant 1/2, \\ 2(1 - x_n) & 1/2 \leqslant x_n \leqslant 1. \end{cases} \tag{1}$$

(a) [5 marks] Sketch the tent map and prove that it defines a map of the unit interval into itself.

(b) [5 marks] Find all fixed points and determine their stability.

(c) [5 marks] Show that there is a single period-2 orbit and analyse its stability.

(d) [5 marks] Show that any initial rational value less than one ends up on a periodic orbit and analyse the stability of this periodic orbit.

(e) [5 marks] The Lyapunov exponent for a map $x_{n+1} = f(x_n)$ is defined as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |f'(x_k)|. \tag{2}$$

Compute the Lyapunov exponent for the tent map and show that it is positive (hence, the system is chaotic).

Section B: Further Mathematical Methods

3. (a) [9 marks] Consider the equation

$$\ddot{x} + (1+\epsilon)x + \epsilon^3 x^3 = \cos t, \qquad x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi).$$

By formally writing

$$x \sim \frac{x_0}{\epsilon} + x_1 + \cdots ,$$

show that

$$x_0 = A\cos t,$$

where A satisfies

$$A + \frac{3A^3}{4} = 1.$$

(b) Consider the equation

$$y(x) = f(x) + \lambda \int_0^1 (x^4 + t^4) y(t)\,\mathrm{d}t$$

where λ is a real constant.

(i) [7 marks] Show that there is a unique solution for $y(x)$ providing $\lambda \neq -15/2$ and $\lambda \neq 15/8$.

(ii) [9 marks] When $\lambda = -15/2$ what is the condition on f for a solution to exist? What is the general solution in this case?

[*You may use the identities*

$$\int_0^{2\pi} \cos^2 t\,\mathrm{d}t = \int_0^{2\pi} \sin^2 t\,\mathrm{d}t = \pi, \quad \int_0^{2\pi} \cos^4 t\,\mathrm{d}t = \int_0^{2\pi} \sin^4 t\,\mathrm{d}t = \frac{3\pi}{4},$$

$$\int_0^{2\pi} \cos t \sin^3 t\,\mathrm{d}t = \int_0^{2\pi} \cos^3 t \sin t\,\mathrm{d}t = 0, \quad \int_0^{2\pi} \cos^2 t \sin^2 t\,\mathrm{d}t = \frac{\pi}{4}.$$

without proof.]

4. Suppose the functions y and z minimise the functional

$$J[y,z] = \int_0^1 F(x,y,\dot{y},z,\dot{z})\,\mathrm{d}x,$$

over all $y, z \in C^2[0,1]$, subject to $y(0) = a$, $z(0) = b$, $y(1) = c$, $z(1) = d$, and the constraint

$$G(y,z) = 0,$$

where F and G are continuously differentiable, and a dot represents $\mathrm{d}/\mathrm{d}x$.

(a) [10 marks] Show that y and z satisfy the Euler equations

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \dot{y}}\right) - \lambda\frac{\partial G}{\partial y} &= 0, \\ \frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \dot{z}}\right) - \lambda\frac{\partial G}{\partial z} &= 0, \end{aligned}$$

where λ is a Lagrange multiplier.

(b) [5 marks] The *Hamiltonian*, H, is given by

$$H = \dot{y}\frac{\partial F}{\partial \dot{y}} + \dot{z}\frac{\partial F}{\partial \dot{z}} - F.$$

Show that

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{\partial F}{\partial x}.$$

(c) An ant is crawling on the outside of the circular cylinder $y^2 + z^2 = 1$. It starts at the point $(x,y,z) = (0,-1,0)$ and takes the shortest path to the point $(x,y,z) = (1,1,0)$.

(i) [8 marks] Find the equation of the path taken, and thus show that it is a helix.

(ii) [2 marks] Find the length of the path.

Section C: Further PDEs

5. (a) [8 marks] Show that an eigenfunction expansion gives the solution of

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = x - \pi, \qquad u(0) = u(\pi) = 0,$$

as

$$u(x) = \sum_{n=1}^{\infty} \frac{2}{n^3} \sin nx.$$

[*You may use without proof the identity* $\int_0^\pi \sin^2 nx \,\mathrm{d}x = \pi/2$.]

(b) The Mellin transform is given by

$$\mathcal{M}[f(x); s] = F(s) = \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x,$$

which exists in some strip $s_1 < \mathrm{Re}(s) < s_2$. The inversion is given by

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s} F(s)\,\mathrm{d}s,$$

where $s_1 < c < s_2$.

(i) [5 marks] Show that for $a > 0$, $\mathcal{M}[f(ax); s] = a^{-s}\mathcal{M}[f(x); s]$. Hence show that if

$$S(x) = \sum_{k=1}^{\infty} \lambda_k g(\mu_k x),$$

then

$$\mathcal{M}[S(x); s] = \Lambda(s)\mathcal{M}[g(x); s], \qquad \text{where} \qquad \Lambda(s) = \sum_{k=1}^{\infty} \lambda_k \mu_k^{-s}.$$

(ii) [12 marks] Using (b)(i) show that

$$\sum_{n=1}^{\infty} \frac{2}{n^3} \sin nx = -\frac{\pi^2 x}{3} + \frac{\pi x^2}{2} - \frac{x^3}{6} + o(x^3)$$

as $x \to 0+$.

[*You may use without proof the fact that the gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t$$

has simple poles at $x = -m$, $m = 0, 1, 2, \ldots$ *with residue* $(-1)^m/m!$. *The Riemann zeta function, defined by*

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$$

for Re$(x) > 1$, *may be analytically continued to a meromorphic function which has a single pole at* $x = 1$ *with residue* 1. *Note also that* $\zeta(0) = -1/2$, $\zeta(2) = \pi^2/6$, *and that* $\mathcal{M}[\sin(x); s] = \sin\left(\frac{\pi s}{2}\right)\Gamma(s)$.]

6. Let the operator L be given by

$$Lu = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$$

on the interval $0 \leqslant x < \infty$ with the conditions $u'(0) = 0$ and $u \in L^2[0, \infty)$.

(a) [10 marks] Show that for μ not a positive real number the Green's function for $Lu - \mu u$ is

$$G(x, \xi; \mu) = \begin{cases} \dfrac{\mathrm{i}}{\sqrt{\mu}} \cos\left(\sqrt{\mu}\, x\right) \mathrm{e}^{\mathrm{i}\sqrt{\mu}\,\xi} & 0 \leqslant x < \xi < \infty, \\ \dfrac{\mathrm{i}}{\sqrt{\mu}} \cos\left(\sqrt{\mu}\, \xi\right) \mathrm{e}^{\mathrm{i}\sqrt{\mu}\,x} & 0 \leqslant \xi < x < \infty, \end{cases}$$

where you should define the branch of the square root.

(b) [10 marks] Use this to find the corresponding spectral representation of the delta function. Show that the cases $x > \xi$ and $x < \xi$ both give the same result.

(c) [5 marks] The Fourier cosine integral transform is defined, for suitable functions $f(x)$, by

$$F(t) = \int_0^\infty \cos(t\xi)\, f(\xi)\, \mathrm{d}\xi.$$

Deduce from (b) that the inversion formula is

$$f(x) = \frac{2}{\pi} \int_0^\infty F(t) \cos(tx)\, \mathrm{d}t.$$

 End of Last Page