Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

Thursday, 19th April 2007, 9:30 a.m. – 11:30 a.m.

Candidates may attempt as many questions as they wish. The best four solutions will count.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Question 1

Use Charpit's method to find solutions (in parametric form) of the first order partial differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + y\frac{\partial u}{\partial y} = u,$$

subject to $u(0, y) = 1 + y \quad \text{for} \quad 0 \leq y \leq 1.$
Explain briefly how to find the region where each of your solutions is determined by the initial data.

Question 2

Define characteristic directions for the system of partial differential equations

$$\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y = \mathbf{c}.$$

Show that $\mathbf{l}^T\mathbf{A}\dot{\mathbf{u}} = \mathbf{l}^T\mathbf{c}\dot{x}$ holds along a characteristic (parameterised by t, with $\dot{x} \equiv dx/dt$, etc), with the corresponding left eigenvector $\mathbf{l}^T$.
Show that, for the system (with $uv \neq 0$)

$$uvv_x + u_y = \frac{1}{2}uv,$$

$$uvu_x + v_y = \frac{1}{2}uv,$$

$u + v - x$ is constant along one characteristic family, and find the corresponding invariant on the other characteristic family.

Question 3

A smooth function $u(x,t)$ in the positive quadrant of the (x,t) plane satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - g(x,t,u) \qquad \text{for} \quad x,t > 0,$$

$$u(x,t) \to 0 \qquad \text{as } x \to \infty, \quad \text{fixed} \quad t > 0,$$

$$u(x,0) = 0, u(0,t) = -1 \qquad \text{for} \quad x,t > 0,$$

where the continuous function $g(x,t,u) > 0$ for $x,t > 0$ and for all u.
Show that $u(x,t)$ cannot have a maximum at any $x_0, t_0 > 0$.
By using similarity solution methods, find $u(x,t)$ when $g(x,t,u) = (2t)^{-1}\exp(-x^2/4t)$ for $x,t > 0$.

Question 4

State the problem satisfied by the Green's function for the boundary value problem

$$\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \text{for} \quad x^2 + y^2 < 1,$$

$$u(x,y) = 0 \qquad \text{for} \quad x^2 + y^2 = 1,$$

and use it to express $u(x,y)$ in terms of an integral involving $f(x,y)$.
If $f(x,y) = f(-x,y)$ for all (x,y), show that your solution satisfies

$$\frac{\partial u}{\partial x} = 0 \text{ on } \{x = 0, -1 \le y \le 1\}.$$

In terms of the above Green's function, find the Green's function for the problem

$$\nabla^2 u = f(x,y) \qquad \text{for} \quad x^2 + y^2 < 1, x > 0,$$

$$u(x,y) = g(x,y) \qquad \text{for} \quad x^2 + y^2 = 1, x > 0,$$

$$\frac{\partial u}{\partial x} = 0 \qquad \text{for} \quad x = 0, -1 \le y \le 1.$$

Question 5

(a) Define a *compact support test function* on $\mathbb{R}$, and a *distribution* on this set of test functions. State the definition of the derivative T' of a distribution T via its action on a test function $\phi(x)$, and hence show that the derivative of the Heaviside function $\mathcal{H}(x)$ is the delta function $\delta(x)$.

(b) Which of the following actions on a test function $\phi(x)$ define distributions (as in part (a))? Give brief reasons.

$$\langle T_1, \phi\rangle = \phi(0)\phi(1);$$
$$\langle T_2, \phi\rangle = \int_{-\infty}^{\infty} (\phi(x) - \phi(-x))\, \mathrm{d}x$$
$$\langle T_3, \phi\rangle = \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(-x)}{x}\, \mathrm{d}x.$$

Identify the distribution T_4 defined by its action as follows:

$$\langle T_4, \phi\rangle = \lim_{\epsilon\to 0} \frac{\phi(\epsilon) - \phi(-\epsilon)}{2\epsilon}.$$

Question 6

The Fourier transform $\hat{T}(k)$ of a distribution T, defined on the set of open support test functions, is defined by its action

$$\langle \hat{T}, \psi\rangle = \langle T, \hat{\psi}\rangle.$$

Assuming standard properties of the Fourier transform of a test function, show that the transform of T' is $-\mathrm{i}k\hat{T}$.

Show that the Fourier transform of $\mathrm{e}^{-a^2x^2}$ is $(\sqrt{\pi}/a)\mathrm{e}^{-k^2/4a^2}$. Explain how to use this result, together with the Fourier inversion formula, to show that $\hat{1} = 2\pi\delta(k)$.

The function $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = u_0(x),$$

where $u_0(x)$ is integrable. Calculate $u(x,t)$ in the form of a convolution integral.