Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

Thursday, 17th April 2008, 9:30 a.m. – 11:30 a.m.

Candidates may attempt as many questions as they wish. The best four solutions will count.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Question 1

Let F be a twice continuously differentiable function of its arguments, and consider the first order partial differential equation

$$F(x, y, u, p, q) = 0, \text{ where } p := u_x,\ q := u_y,$$

together with the initial data $\Gamma : x_0(s), y_0(s), u_0(s)$ for $0 \leq s \leq 1$. Briefly explain Charpit's Method for finding solutions $u(x, y)$ (possibly in parametric form) of this problem, stating clearly where you would expect such solutions to be determined by the data Γ.

Solve

$$x\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)^2 - u - 1 = 0,$$

where $u(x, 0) = x$ for $0 \leq x \leq 1$, indicating on a sketch where your solutions are determined by the initial data.

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Question 2

Suppose that $v > 0$ and that $c(v) > 0$, and that u and v satisfy the partial differential equations

$$u_x + uu_y + \frac{c^2(v)}{v} v_y = v\,,$$
$$v_x + uv_y + vu_y = -\frac{v^2}{c(v)}\,.$$

Show that the system is hyperbolic, and that the characteristics are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u \pm c(v).$$

Along the positive characteristics, $y' = u + c(v)$, show that the Riemann invariant

$$u + \int^v \frac{c(\hat{v})}{\hat{v}}\,\mathrm{d}\hat{v}$$

is constant.

Suppose that the initial data is $u(0, y) = c(y), v(0, y) = y$ for $1 < y < \infty$.

By evaluating the Riemann invariant on the positive characteristics, show that, if $c(v) = 1/v$, then $u = c(v)$ throughout the region which they span. In this case, show that the negative characteristics are straight lines there, and by integrating the characteristic equation along these straight lines show that the solution in this region is

$$u = 1/v = \sqrt{2x + 1/y^2}.$$

Question 3

The twice continuously differentiable function $u(x,t)$ satisfies

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t,u) && \text{for } x,t>0,\\
&u(x,t)\to 0 \ \text{ as } \ x\to\infty && \text{for fixed } t>0,\\
&u(x,0)=0,\ u(0,t)=1 && \text{for } x,t>0,\\
&u(x,t) \ \text{ is bounded} && \text{for } x,t\ge 0,
\end{aligned}$$

where the continuous function $f(x,t,u)$ satisfies $f(x,t,u)<0$ for $x,t>0$, and $f\to 0$ as $1/x, u\to 0$.

(a) Show that $u(x,t)$ cannot have a maximum at any $x_0, t_0 > 0$.

(b) State the conditions that a Green's function $G(x,t,\xi,\eta)$ for the above problem must satisfy in order that $u(\xi,\eta)$ may be expressed in terms of integrals involving G and f; and hence derive such an expression.

(c) Find $u(x,t)$ when $f(x,t,u) = (2t)^{-1}\exp(-x^2/4t)$.

[*Hint: find $u(x,t)$ by using a similarity solution of the form $F(\eta)$, where $\eta = x/t^{\beta}$.*]

Question 4

Let f be a continuous function in the domain $D \subset \mathbb{R}^2$, and let $u(\underline{x}) = u(x,y)$ satisfy

$$\begin{aligned} -\Delta u &= f(x,y) \quad \text{for } (x,y) \in D, \\ u(x,y) &= 0 \quad \text{for } (x,y) \in \partial D \text{ (the boundary of } D). \end{aligned} \tag{P}$$

State the defining properties for a suitable Green's function $G(\underline{x}, \underline{\xi})$, where $\underline{x}, \underline{\xi} \in \overline{D}$ for this problem (P), and hence briefly derive the following expression for the solution of problem (P):

$$u(\underline{\xi}) = \iint_D G(\underline{x}, \underline{\xi}) f(\underline{x}) \, \mathrm{d}\underline{x} \quad \text{for } \underline{\xi} \in D.$$

Let the orthonormal sequence $\{\phi_n\}, n = 1, 2, \ldots$, satisfy

$$\begin{aligned} -\Delta \phi_n &= \lambda_n \phi_n \quad \text{for } (x,y) \in D, \\ \phi_n &= 0 \quad \text{for } (x,y) \in \partial D, \end{aligned}$$

where $\lambda_n \in \mathbb{R}$ form an ordered sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots$.
Show that, if $\{\phi_n\}$ is a suitable spanning set so that

$$G(\underline{x}, \underline{\xi}) = \sum_{n=1}^{\infty} c_n(\underline{\xi}) \phi_n(\underline{x}),$$

then

$$G(\underline{x}, \underline{\xi}) = \sum_{n=1}^{\infty} \phi_n(\underline{x}) \phi_n(\underline{\xi}) / \lambda_n \ .$$

When $D = \{(x,y) : \ x^2 + y^2 < 1\}$, find $G(\underline{x}, \underline{\xi})$ explicitly.
If also $f(x,y) = (x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4$, find $u(x,y)$ explicitly.
[*Hint:* $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$ *when* $r^2 := x^2 + y^2$, $\theta = \tan^{-1} y/x$ *are polar co-ordinates.*]

Question 5

(a) Use the language of distributions to rigorously define the delta function $\delta_\xi \equiv \delta(x - \xi)$ with singularity $\xi = (\xi_1, \xi_2, \xi_3)$, where $\xi \in \Omega$ and Ω is a subset of 3-dimensional Euclidean space $\mathbb{R}^3$. Be sure to explicitly specify and describe the proper space in which each function lives.

(b) Show using distributions that

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

for $x \in \mathbb{R}$ and $a \neq 0$. Use this result to verify that $\delta(x)$ is even.

(c) Give a rigorous definition of what it means for a sequence of scalar functions $f_n(x)$ ($x \in \mathbb{R}$) to converge weakly as $n \to \infty$. Show that the function $f_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}$ converges weakly, with

$$\lim_{n \longrightarrow \infty} \int_{-\infty}^{\infty} f_n(x) g(x) dx = g(0)\,,$$

for any test function g.

(d) Show that for a continuous scalar function f (which is not necessarily differentiable), the expression $u = f(x - ct)$ (where $x \in \mathbb{R}$ and t is time) is a weak solution of the partial differential equation

$$u_t + cu_x = 0\,.$$

Question 6

Consider the viscous Burgers equation,

$$\begin{aligned} u_t + uu_x &= \epsilon u_{xx}\,, \\ u(x,0) &= u_0(x)\,. \end{aligned} \tag{1}$$

(a) Use the method of characteristics to show that for $\epsilon = 0$, one can find $u(x,t)$ by satisfying the equation

$$F(u) \equiv u_0(x - tu) - u = 0\,. \tag{2}$$

(b) If $\epsilon = 0$ and $u_0(x) = \sin x$, use the result of part (a) to find the critical time $t_c \geq 0$ at which solutions first become singular (i.e., the earliest time that a shock occurs).

(c) Let $\epsilon = 0$ still hold and consider the "Riemann problem"

$$u_0(x) = \begin{cases} u_l\,, & x \leq 0\,, \\ u_r\,, & x > 0\,. \end{cases} \tag{3}$$

Draw the characteristics in the (x,t)-plane for all possible situations. When do shocks occur?

(d) Now consider the case $\epsilon > 0$ with $u_0(x)$ given by (3) with $u_l > u_r$. Let $\xi = (x - st)/\epsilon$ and derive a formula for "traveling wave solutions" $u = w(\xi)$ of (1,3) with propagation speed s. Give an expression for the weak solution u that one obtains in the limit $\epsilon \to 0$. (You do *not* have to prove that it's a weak solution. Just derive the formula.) Also give an expression for the propagation speed of the solution obtained in this limit.