JMAT 7303

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

TRINITY TERM 2011
Thursday, 28nd April 2011, 9:30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Applied Partial Differential Equations

Question 1

Find, in parametric form, the solution of the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(x\mathrm{e}^{u}\right) = 0$$

in $t > 0$, subject to the initial condition $u = u_0(x)$ at $t = 0$.

[10 marks]

For the case where $u_0(x) = -2\log(x)$ for $x \geq 1$, find $u(x,t)$ in explicit form.

[8 marks]

Determine the region of the (x,t)-plane where the solution is uniquely defined by the data, and sketch this region.

[7 marks]

Question 2

Show that the system

$$\begin{aligned}\frac{\partial u}{\partial t} + u^2 \frac{\partial v}{\partial x} &= 0,\\ \frac{\partial v}{\partial t} + w \frac{\partial u}{\partial x} &= 0,\\ \frac{\partial w}{\partial t} + 2uw \frac{\partial v}{\partial x} &= 0\end{aligned}$$

is hyperbolic if $u \neq 0$ and $w > 0$.

[6 marks]

Assuming both of these to be true, find the characteristics and identify any Riemann invariants.

[10 marks]

Suppose the initial conditions

$$u = f(x), \qquad v = f(x), \qquad w = f(x)^2$$

are given at $t = 0$. Show that $u(x, t)$ satisfies the implicit equation

$$u = f\left(x - u^2 t\right)$$

for $t > 0$, and find corresponding expressions for v and w.

[9 marks]

Question 3

Classify the partial differential equation

$$2\frac{\partial^2 u}{\partial x^2} + 2\left(\mathrm{e}^x - 1\right)\frac{\partial^2 u}{\partial x \partial y} + \left(1 + \mathrm{e}^{2x}\right)\frac{\partial^2 u}{\partial y^2} + \frac{2\mathrm{e}^x}{1+\mathrm{e}^x}\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right) - \left(1 + \mathrm{e}^x\right)^2 u = 0.$$

[5 marks]

Show that $\xi = \left(x + \mathrm{e}^x\right)/2$ and $\eta = y + \left(x - \mathrm{e}^x\right)/2$ are canonical variables and reduce the equation to canonical form.

[10 marks]

Now suppose that $u(\xi, \eta)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 2u$$

in $\eta > 0$ and the boundary conditions

$$u = f(\xi) \text{ on } \eta = 0, \qquad u \to 0 \text{ as } \eta \to \infty, \qquad u \to 0 \text{ as } \xi \to \pm\infty.$$

By taking a Fourier transform in ξ (or otherwise), show that the solution may be written in the form

$$u(\xi, \eta) = \int_{-\infty}^{\infty} f(s) g(\xi - s, \eta)\,\mathrm{d}\eta, \qquad \text{where} \quad g(\xi, \eta) = \frac{1}{\pi}\int_0^{\infty} \mathrm{e}^{-\eta\sqrt{k^2+2}} \cos(k\xi)\,\mathrm{d}k.$$

[10 marks]

Question 4

Write down the conditions satisfied by the *Riemann function* $R(x, y; \xi, \eta)$ for the linear hyperbolic partial differential equation

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x \partial y} + a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u = f(x, y).$$

[3 marks]

If the Cauchy data $u = u_0(x)$, $u_x = p_0(x)$ are specified on the line $y = -x$, show that the solution may be written in the form

$$u(\xi, \eta) = \iint_D Rf \,\mathrm{d}x\mathrm{d}y + R(-\eta, \eta; \xi, \eta)u_0(-\eta) + \int_A^B \Big\{u_0\big((a + b)R - R_y\big) + p_0 R\Big\}\,\mathrm{d}x,$$

carefully defining the points A and B and the domain of integration D.

[15 marks]

Find the Riemann function for the equation

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} + xu = 0.$$

[*Hint: you may find it helpful to consider the equation satisfied by* $F = R_x + xR$.]

[7 marks]

Section B — Further Applied Partial Differential Equations

Question 5

Define the Hankel Transform and its inverse for a function $u(r, t)$ with respect to the variable r. **[4 marks]**

Show how the inverse Hankel Transform can be derived from the double Fourier Transform. **[15 marks]**

The radially symmetric function $u(r, z)$ satisfies Laplace's equation with respect to the cylindrical polar co-ordinates (r, z). Show that the Hankel transform $H[u]$ satisfies the differential equation

$$\frac{\partial^2 H[u]}{\partial z^2} - k^2 H[u] = 0.$$

[6 marks]

[*You may quote the result* $J_n(kr) = \frac{1}{2\pi} \int_{\phi_0}^{2\pi+\phi_0} \mathrm{d}\phi \exp(i(n\phi - kr\sin\phi))$].

Question 6

(a) Find travelling wave solutions of the form $u(x,t) = f(x-ct)$ for Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

subject to $u \to 0$ as $x \to +\infty$ and $u \to u_0\,(>0)$ as $x \to -\infty$. **[12 marks]**

(b) Use the transformations

$$\begin{aligned} \frac{1}{2}(u_x + v_x) &= a\sin\left(\frac{u-v}{2}\right) \\ \frac{1}{2}(u_t - v_t) &= \frac{1}{a}\sin\left(\frac{u+v}{2}\right) \end{aligned} \qquad (1)$$

to show that u and v satisfy the Sine-Gordon equations **[8 marks]**

$$u_{xt} = \sin u, \quad v_{xt} = \sin v.$$

When $v = 0$, show that (1) admits the solution $u(x,t) = 4\arctan C\exp(ax + \frac{t}{a})$ for some arbitrary constant C. **[5 marks]**