JMAT 7303

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

TRINITY TERM 2011
Thursday, 19 April 2012, 9:30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Applied Partial Differential Equations

Question 1

Consider the first order partial differential equation for $u(x,y)$

$$F(p,q,u,x,y)=0, \quad \text{where } p=\frac{\partial u}{\partial x}, \ \ q=\frac{\partial u}{\partial y},$$

and F is C^2 in its arguments. Suppose further that u is specified along the curve $\Gamma(x,y)$ so that

$$x=x_0(s), \ \ y=y_0(s), \ \ u=u_0(s) \quad \text{on } \ \Gamma(x,y).$$

(a) State Charpit's equations for solving this problem. State also appropriate initial data for their solutions. Show that $F=0$ along their solutions.

[8 marks]

(b) Find a solution in parametric form for

$$p^2+pq+u-1=0,$$

with boundary data $u(x,0)=x^2/8$. Find the region in the (x,y)-plane in which your solution is determined by the boundary data.

[17 marks]

Question 2

(a) Suppose that $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \tag{1}$$

in the rectangular domain $\mathcal{D} = \{(x,t) : x_1 < x < x_2,\ 0 < t < \tau\}$. Show that u cannot attain a maximum inside $\mathcal{D}$ or on the line $t = \tau$ when $f(x,t) \leq 0$. Hence show that there is at most one solution to (1) that satisfies the following initial and boundary conditions

$$\left.\begin{array}{lll} u(x,0) & = u_0(x), & x_1 < x < x_2, \\ u(x_1,t) & = u_1(t), & 0 < t < \tau, \\ u(x_2,t) & = u_2(t), & 0 < t < \tau. \end{array}\right\}$$

[10 marks]

(b) Suppose now that $u(x,t)$ satisfies the problem

$$\left.\begin{array}{lll} \dfrac{\partial u}{\partial t} & = \dfrac{\partial}{\partial x}\left(u^2 \dfrac{\partial u}{\partial x}\right) & 0 < x < t^\theta, \\ \dfrac{\partial u}{\partial x}(0,t) & = 0, & t > 0, \\ u(x,t) & = 0, & \text{on } x = t^\theta, \end{array}\right\} \tag{2}$$

where $u(x,t) > 0$ for $0 < x < t^\theta$. Determine the value of $\theta > 0$ and the function $f \in C^2$ for which (2) admits a similarity solution of the form

$$u(x,t) = \frac{1}{t^\theta} f(\eta), \qquad \text{where } \eta = \frac{x}{t^\theta}.$$

[15 marks]

Question 3

The continuously differentiable functions $h(x,t)$ and $u(x,t)$ satisfy the shallow water equations:

$$\left.\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + \left(hu^2 + \frac{h^2}{2}\right)_x &= 0. \end{aligned}\right\} \tag{3}$$

(a) Show that this system is hyperbolic for $h > 0$ and determine the characteristic curves in the (x,t) plane. Show further that the quantities $u \pm 2\sqrt{h}$ are preserved along the characteristic curves.

[10 marks]

(b) Suppose that the n-dimensional system

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{\partial \mathbf{Q}}{\partial x} = 0.$$

possesses a shock at $x = X(t)$. State the Rankine-Hugoniot conditions for the shock speed.

Explain why the shock is causal if $(n-1)$ families of characteristic curves leave the shock and $(n+1)$ families of characteristics propagate into it.

[7 marks]

(c) Suppose now that equations (3) admit a shock solution and denote by $u_\pm$ and $h_\pm$ the values of u and h ahead of (+) and behind (-) the shock. Using the results from parts (a) and (b), show that the shock is causal if $h_+ < h_-$.

[8 marks]

Question 4

(a) Consider the problem for $u(x, y)$

$$\left.\begin{aligned} \nabla^2 u &= f(x,y), \qquad & (x,y) \in \mathcal{D}, \\ \alpha u + \frac{\partial u}{\partial n} &= g(x,y), \qquad & (x,y) \in \partial\mathcal{D}, \end{aligned}\right\} \tag{4}$$

in which α, f and g are continuously differentiable functions of (x, y), $\mathcal{D}$ is a region of the (x, y)-plane bounded by the simple, smooth, closed curve $\partial\mathcal{D}$, and $\partial/\partial n$ denotes the outward normal derivative on $\partial\mathcal{D}$. Prove that there is at most one solution when $\alpha > 0$. Show further that there is no solution when $\alpha = 0$ unless a solvability condition (which you should determine) is satisfied. Show that when this condition is satisfied the solution is non-unique.

[8 marks]

(b) You are given that $G(x, y; \xi, \eta)$ solves

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(\mathbf{x} - \boldsymbol{\xi}) - \alpha, \qquad (x,y) \in \mathcal{D},$$

$$\frac{\partial G}{\partial n} = 0, \qquad (x,y) \in \partial\mathcal{D}.$$

where $\mathbf{x} = (x, y)$, $\boldsymbol{\xi} = (\xi, \eta)$. Determine the constant α for which G exists.

Show further that for $\boldsymbol{\xi} \in \mathcal{D}$ the solution to (4) is

$$u(\xi, \eta) = \bar{u} + \iint_{\mathcal{D}} G f \, dx dy - \int_{\partial\mathcal{D}} G g \, ds,$$

where $\bar{u}$ is the average value of u in $\mathcal{D}$.

[7 marks]

(c) Suppose that $u(r, \theta)$ solves

$$\nabla^2 u = 0, \qquad \text{for } 1 < r^2 < a^2,$$

with

$$\frac{\partial u}{\partial r} + \alpha_1 u = \cos 2\theta \ \text{ on } r = 1 \qquad \text{and} \qquad \frac{\partial u}{\partial r} + \alpha_2 u = 0 \ \text{ on } r = a,$$

where α_1, α_2 are constants.

Use separation of variables to construct a solution for $u(r, \theta)$. Determine the curve in (α_1, α_2) parameter space on which the solution is not unique.

[10 marks]

Section B — Further Applied Partial Differential Equations

Question 5

(a) Write down the forward Hankel transform of a function $T(r, z)$ with respect to r, and the corresponding inverse transformation. **[4 marks]**

(b) Derive the expression for the forward Hankel transform from the two-dimensional Fourier transform

$$\tilde{T}(\mathbf{k}) = \iint T(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\ \mathrm{d}x\mathrm{d}y.$$

[6 marks]

(c) Suppose that the half-space $z \geq 0$ contains a material with thermal conductivity κ. A total heat flux Q is applied uniformly over the surface of the disc $r \leq a$, while the remainder of the $z = 0$ surface is insulated. The steady-state temperature distribution is given by the solution of $\nabla^2 T = 0$ with these boundary conditions. Show that the solution that decays to zero as $z \to +\infty$ may be written as

$$u(r, z) = \frac{Q}{\pi\kappa a}\int_0^\infty \frac{J_0(kr)J_1(kr)}{k}\mathrm{e}^{-kz}\,\mathrm{d}k.$$

[*You may use the integral representation* $J_n(x) = (2\pi)^{-1}\int_0^{2\pi}\exp\{i(n\phi - x\cos\phi)\}\mathrm{d}\phi$ *and the relation* $J_1'(r) = J_0(r) - r^{-1}J_1(r)$.] **[15 marks]**

Question 6

(a) Show that the operators

$$L = \begin{pmatrix} \partial_x & (1/2)u_x \\ (1/2)u_x & -\partial_x \end{pmatrix}, \quad M = \frac{1}{4k}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix},$$

constitute a Lax pair for the sine–Gordon equation $u_{xt} = \sin u$. **[10 marks]**

(b) Now change variables to $X = (t+x)/2$ and $T = (t-x)/2$. Use the substitution $u = 4\tan^{-1} F(X,T)$ in the sine–Gordon equation to derive the equation

$$(1+F^2)\left(F_{TT} - F_{XX} + F\right) - 2F\left(F_T^2 - F_X^2 + F^2\right) = 0,$$

[10 marks]

and show that $F(X,T) = T \operatorname{sech} X$ is a solution. **[5 marks]**

[*You may use the relation* $\sin 4\theta = 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta$.]